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THE CAUCHY PROBLEM FOR THE VLASOV-DIRAC-BENNEY EQUATION AND RELATED ISSUES IN FLUID MECHANICS AND SEMI-CLASSICAL LIMITS

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Abstract. This contribution concerns a one-dimensional version of the Vlasov equation dubbed the Vlasov−Dirac−Benney equation (in short V−D−B) where the self interacting potential is replaced by a Dirac mass. Emphasis is put on the relations between the linearized version, the full nonlinear problem and equations of fluids. In particular the connection with the so-called Benney equation leads to new stability results. Eventually the V−D−B appears to be at the "cross road" of several problems of mathematical physics which have as far as stability is concerned very similar properties.

1. **Introduction.** This article is devoted to a one-dimensional variant of the classical Vlasov-Poisson equation where the Coulomb interacting potential V is replaced by the Dirac mass δ:

$$\begin{aligned}&\partial_t f(t,x,v) + v\partial_x f(t,x,v) - \partial_x \rho_f(t,x)\partial_v f(t,x,v) = 0,\\ &\rho_f(t,x) = \int_{\mathbb{R}} f(t,x,v)dv.\end{aligned} \tag{1}$$

Since, in one of the most important configuration it is equivalent to the Benney equation (cf. section 6 and [3]) we call it the Vlasov−Dirac−Benney equation or in short V−D−B.

This equation exhibits both some similarities (at the level of the formal structures) and some basic differences with generalizations of the following Vlasov–Poisson equation:

$$\begin{aligned}&\partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f = 0,\\ &E = -\nabla_x \int_{\mathbb{R}^d} V(x-y)\Big(\int_{\mathbb{R}^d} f(t,y,v)dv - 1\Big)dy,\end{aligned} \tag{2}$$

where the self-interacting electric field E is deduced from the density f of particles through the action of a potential V which may differ from the original Coulomb

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potential and may be more or less regular than this potential. Especially in this paper we focus on the case where the potential V is the Dirac distribution δ.

On one hand all the equations of the type (2) share in common some essential properties recalled below.

They are "Liouville equations" associated to a dynamical flow defined by the equations

$$\dot{x}(t) = v(t)\,, \quad \dot{v}(t) = -\int_{\mathbb{R}^d} \nabla_x V(x(t)-y)\Big(\int_{\mathbb{R}^d} f(t,y,w)dw - 1\Big)dy.$$

They (at least formally) conserve the energy

$$\begin{aligned}\mathcal{E}(f) = &\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|v|^2}{2} f(t,x,v)dxdv \\ &+ \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} V(x-y)f(t,x,v)f(t,y,w)dwdydxdv.\end{aligned} \tag{3}$$

On the other hand the uniform background which is represented by the constant 1 in the definition of the global density $\rho = \int_{\mathbb{R}^d} f(t,y,v)dv - 1$, to ensure global neutrality of the plasma, can be removed (since the potential V only appears through its gradient in (2)) and thus when the potential V is a Dirac mass the equation becomes

$$\partial_t f + v\cdot\nabla_x f - \nabla_x \rho_f \cdot \nabla_v f = 0, \quad \rho_f(x,t) = \int_{\mathbb{R}_v} f(t,x,v)dv. \tag{4}$$

Moreover in (4) the mapping $f \mapsto \rho_f \mapsto E = -\nabla_x \rho_f$ is an operator of degree 1 while for the original Vlasov–Poisson equation it is an operator of degree -1. Therefore the effect of the instabilities will be much more drastic and while for the original Vlasov–Poisson equation the issue is the large time asymptotic behavior, here the issue is that the Cauchy problem may be badly posed even for regular initial data and for arbitrarily small time.

Below we focus on the equation (1) which is the one-dimensional ($d=1$) version of the problem and where instead of the symbol ∇ the symbol ∂ is used.

The interest of a one-dimensional space model may be fully justified by physical reasons, particularly in the quasineutral-limit when the Debye length vanishes (cf [16]). Moreover it is in one dimension that the spectral analysis of the linearized problem is, by an adaptation of the method of Penrose (cf [36]), the most explicit. Then, as we try to show, there is a natural connection between the properties of the linearized and the fully nonlinear model. This connection emphasizes the role of "bumps" in the initial profile. In particular in the case of the one-bump profile the connection with the Benney equation gives a new stability theorem for the full nonlinear problem. Eventually the stability results are in full agreement with what is known concerning the WKB limit of the Non-Linear Schrödinger equation.

This article is a follow up of two contributions of the authors [2] *where the first systematic analysis of instabilities for the* $V-D-B$ *was established and* [4] *where (with the introduction of integral Riemann invariants with value in an infinite dimensional space) first results for the Cauchy problem were proven. As such it is a small contribution to research activities in Kinetic theory and Vlasov equation. In this field Seiji Ukai, starting around 67 and up to the very last moment of life produced essential breakthroughs containing fundamental tools for our community. Therefore we hope that the subject of our small contribution will be well adapted in this volume devoted to his Memory. The first author had the chance to meet Seiji* *already in 1974 and since this period was always fascinated, not only by is creativity in Mathematical Science but also by his sense of humour, friendship and generosity.*

2. The modal analysis of the linearized equation.

In this section one considers in $\mathbb{R}_x \times \mathbb{R}_v$ the dynamics of the V−D−B equation for fluctuations near the neighborhood of space independent probability distribution function $G(v)$,

$$G(v) \geq 0\,, \quad \int_{\mathbb{R}} G(v)dv = 1,$$

which is an obvious stationary solution of the equation

$$\partial_t f + v\partial_x f - \partial_x \rho_f \partial_v f = 0, \quad \rho_f(t,x) = \int_{\mathbb{R}_v} f(t,x,v)dv.$$

Therefore f is changed into $G(v) + f$ (f denoting now the fluctuations) and the linearized equation becomes

$$\partial_t f + v\partial_x f - \partial_x \rho G'(v) = 0,$$

by dropping quadratic terms in the fluctuations.

2.1. Modal analysis of the linearized equation.

A standard tool in the analysis of the general (for any potential V) Vlasov equation is the introduction of modal analysis, i.e. solutions (whenever they exist) of the form

$$e_k(t,x,v) = A(k,v)e^{i(kx-\omega(k)t)},$$

leading to the equation

$$(-i\omega(k) + ikv)A(k,v) - ik\widehat{V}(k)\hat{\rho}_A(k)G'(v) = 0,$$

which upon integration with respect to v is equivalent to

$$A(k,v) - \widehat{V}(k)\frac{G'(v)}{v - \omega(k)/k}\hat{\rho}_A(k) = 0, \tag{5}$$

and

$$\left(1 - \widehat{V}(k)\int_{\mathbb{R}} \frac{G'(v)}{v-\omega(k)/k}dv\right)\hat{\rho}_A(k) = 0. \tag{6}$$

Therefore instabilities appear whenever there exists a solution $\omega(k)$ of the above system with

$$\operatorname{Im}\omega(k) > 0\,,$$

where Im denotes the imaginary part. This approach has been developed by Penrose [36] for the one-dimensional Vlasov-Poisson equation where one has $\widehat{V}(k) = 1/k^2$. In this case the system (5) and (6) has no solution for $\operatorname{Im}\omega \neq 0$ and $|k|$ large enough. This is in agreement with the fact that the problem is always well posed and that the only issues are its asymptotic behavior for $|t|$ going to ∞.

Below we adapt the Penrose construction to the linearized V−D−B equation

$$\partial_t f + v\partial_x f - \partial_x \rho_f G'(v) = 0, \quad \rho_f(t,x) = \int_{\mathbb{R}} f(t,x,v)dv.$$

The main difference is that now the operator

$$f \mapsto \partial_x \rho_f G'(v),$$

is "of order 1" while for the Vlasov-Poisson equation it was "of order −1" and therefore it supports more violent perturbations. With $\omega(k) = \omega^* k$ (observe that ω^* has the dimension of a complex velocity and note the link with the typical dispersion relation for the frequency of acoustic waves $\mathrm{Re}\,\omega = ck$, with c a real constant velocity) the system (5) and (6) becomes

$$A(k,v) - \frac{G'(v)}{v-\omega^*}\hat{\rho}_A(k) = 0, \tag{7}$$

and

$$\left(1 - \int_{\mathbb{R}_v} \frac{G'(v)}{v-\omega^*}dv\right)\hat{\rho}_A(k) = 0. \tag{8}$$

We now introduce the open sets $\Im_\pm = \{\omega \in \mathbb{C}, \pm\mathrm{Im}\,\omega > 0\}$, the mapping

$$Z : \omega \mapsto Z(\omega) = \int_{\mathbb{R}_v} \frac{G'(v)}{v-\omega}dv\,,$$

and call solutions ω of the equation

$$Z(\omega) = 1 \quad \text{with} \quad \mathrm{Im}\,\omega \neq 0,$$

unstable modes. Observe that $\omega \in \Im_+$ is a solution of (8) if and only $\overline{\omega} \in \Im_-$ is also a solution. Therefore it is enough to consider only the set $Z(\Im_+)$. As in [36] one has the

Proposition 1. *Assume for the probability profile $v \mapsto G(v)$ the following regularity hypothesis:*

$$v \mapsto G'(v) \in \mathscr{C}^{0,\alpha}(\mathbb{R}) \cap L^1(\mathbb{R})\,. \tag{9}$$

Then the mapping $\omega \mapsto Z(\omega)$ is well defined and analytic on $\Im_+$. Moreover $Z(\Im_+)$ is a bounded set with boundary given by

$$\partial(Z(\Im_+)) = \left\{ w \in \mathbb{R} \mapsto \mathrm{p.v.}\int_{\mathbb{R}} \frac{G'(v)}{v-w}dv + i\pi G'(w) \right\};$$

$\partial(Z(\Im_+))$ is a bounded curve which go to 0 for $w \to \pm\infty$.

Therefore the existence or non-existence of unstable modes is equivalent to the fact that 1 belongs or not to the set $Z(\Im_+)$. Proceeding as in [36] one follows the curve

$$\partial Z_+ : \omega \in \mathbb{R} \mapsto \mathrm{p.v.}\int_{\mathbb{R}} \frac{G'(v)}{v-w}dv + i\pi G'(w).$$

This curve starts and ends at the origin for $\omega = \pm\infty$ and winds round $Z(\Im_+)$ anticlockwise.

To have $1 \notin Z(\Im_+)$ it is sufficient that any $Z(v^*)$ point of intersection of ∂Z_+ (whenever it exists) belongs to the interval $]-\infty, 1[$. However to have $1 \in Z(\Im_+)$ it is necessary (but not sufficient, at variance with the standard Penrose Criteria) that ∂Z_+ crosses the real axis at a point $Z_+(v^*) \in [1, \infty)$. This leads to the following

Theorem 2.1. *Assume for the profile G the regularity hypothesis (9) then*

1. If for any solution of $v^* \in \mathbb{R}$ of the equation $G'(v^*) = 0$ one has

$$\mathrm{p.v.}\int_{\mathbb{R}} \frac{G'(v)}{v - v*}dv < 1, \tag{10}$$

there are no unstable modes.

2. If $G(v)$ has a minimum v^* with the relation

$$\mathrm{p.v.}\int_{\mathbb{R}} \frac{G'(v)}{v-v^*}dv > 1,$$

and no maximum with v^{**} *with*

$$\text{p.v.} \int_{\mathbb{R}} \frac{G'(v)}{v - v^{**}} dv > 1,$$

there exist unstable modes.

Proof. For the proof it is enough to follow the curve ∂Z_+ from $\omega^* = -\infty$ to $\omega^* = +\infty$. ☐

Remark 1. 1. The above theorem concerns points $v^* \in \mathbb{R}$ where $G'(v^*) = 0$, therefore (with the regularity hypothesis (9)), the Cauchy principal values of integrals, denoted by "p.v.", are in fact classical integrals.

2. For any ω^* with $\operatorname{Im} \omega^* \neq 0$ the validity of the integration by part

$$\int_{\mathbb{R}} \frac{G'(v)}{v - \omega^*} dv = \int_{\mathbb{R}} \frac{G(v)}{(v - \omega^*)^2} dv, \tag{11}$$

implies that the relation (8) is well defined not only for profiles in the class $\mathscr{C}^{1,\alpha}$ but also in $\mathcal{M}_b$, the set of finite or bounded Radon measures. As a consequence the statements of the Theorem 2.1 can be extended to more general profiles. For instance if the profile G is the limit in the weak-$*$ topology $\sigma(\mathcal{M}_b, \mathscr{C}_b)$ (the topology of narrow convergence) of a family of probabilities $G_\epsilon(v) \in \mathscr{C}^{1,\alpha}$ satisfying the property

$$\forall \omega^* \in \Im_+, \qquad \left| 1 - \int_{\mathbb{R}} \frac{G'_\epsilon(v)}{v - \omega^*} dv \right| > \eta, \tag{12}$$

with η independent of ϵ there are no unstable modes.

3. Let us note that the Hilbert transform $f \mapsto \mathscr{H} f = \text{p.v.}(1/v) * f$, appearing in the Penrose like criterion (cf. Theorem 2.1 Eq. (10)) defines a continuous endomorphism on several spaces more or less regular, typically: space of entire functions, Hölder spaces $\mathscr{C}^{m,\alpha}$, $(m > 0)$ [33], Hardy spaces H^p, $(p > 0)$ [38] and Lebesgue spaces L^p, $(1 < p < \infty)$ [37, 42]. It can be extended to a continuous map from L^∞ (resp. L^1) to BMO, the space of functions having bounded mean oscillations, (resp. weak L^1 or Marcinkiewicz-Lorentz $L^{1,\infty}$ spaces) [38]. Moreover since $\text{p.v.}(1/v) \in \mathcal{S}'$ (tempered distributions) $\mathscr{H}$ maps $\mathscr{E}'$ (compactly supported distributions) to $\mathcal{S}'$ [17]. It is also possible to extend $\mathscr{H}$ to other spaces of distribution [35]. Nevertheless since the Penrose like criterion involves a pointwise value condition ($G'(v^*) = 0$) the modified Penrose criterion (12) is required.

As explained in classical books of plasma physics (cf. [28] Chapter 9) longitudinal electrostatic kinetic instabilities are related to the effect of "bump-on-tail" in the profile $G(v)$. This is in agreement with the following examples where the existence of unstable modes, is discussed either as illustration of the Theorem 2.1 or with direct computations.

Example 1. A profile $G(v) \in \mathscr{C}^{1,\alpha}$ with only one local maxima (say v^*) generates no unstable mode. In fact for the only point where ∂Z_+ crosses the real axis is v^* and since $G(v)$ is increasing for $v < v^*$ and decreasing for $v > v^*$ one has

$$\int_{\mathbb{R}} \frac{G'(v)}{v - v^*} dv < 0 < 1.$$

Example 2. In particular when $G(v) = \delta_v$ is a Dirac mass (the extreme case of one simple bump) there is no unstable mode. With the point 2 of the Remark 1 this follows from the point 1 of the Theorem 2.1. Moreover this can also be proven by the following explicit computation

$$G(v) = \delta_v \Longrightarrow \int_{\mathbb{R}} \frac{G'(v)}{v-\omega} dv = \int_{\mathbb{R}} \frac{\delta_v}{(v-\omega)^2} dv = \frac{1}{\omega^2},$$

and therefore the solutions of the dispersion equation $\omega^2 = 1$ (cf. Eq. (8)) are given by $\omega^* = \pm 1$, numbers with no imaginary part.

Example 3. However for $G(v) = \frac{1}{2}(\delta_{v-a} + \delta_{v+a})$ the existence of unstable modes depends on the size of a. Dirac masses generate unstable modes, if and only if they are close enough, according to the formula

$$1 - \int_{\mathbb{R}} \frac{G'(v)}{v-\omega} dv = 1 - \frac{1}{(a-\omega)^2} + \frac{1}{(a+\omega)^2},$$

which has non real solutions if and only if $a^2 < 2$.

Example 4. Assume that $G(v)$ is even with $G(0) = G'(0) = 0$, then for ϵ small enough,

$$G_\epsilon(v) = \frac{1}{\epsilon} G\left(\frac{v}{\epsilon}\right),$$

generates unstable modes. In fact for $\omega^* = i\sigma$ with $\sigma \in \mathbb{R}$ the solution of the equation

$$1 - \int_{\mathbb{R}} \frac{G'_\epsilon(v)}{v-\omega^*} dv = 0,$$

becomes

$$\begin{aligned} 0 = 1 - \int_{\mathbb{R}} \frac{G'_\epsilon(v)}{v-\omega^*} dv &= 1 - \int_{\mathbb{R}} \frac{G'_\epsilon(v)v}{v^2+\sigma^2} dv - i \int_{\mathbb{R}} \frac{G'_\epsilon(v)\sigma}{v^2+\sigma^2} dv \\ &= 1 - \int_{\mathbb{R}} \frac{G'_\epsilon(v)v}{v^2+\sigma^2} dv\,. \end{aligned} \tag{13}$$

Eventually the function

$$\sigma \mapsto I(\sigma) = \int_{\mathbb{R}} \frac{G'_\epsilon(v)v}{v^2+\sigma^2} dv,$$

is continuous decreasing from $I(0) = \int_{\mathbb{R}} \frac{G'_\epsilon(v)v}{v^2} dv$ to $I(\infty) = 0$ and by continuity the existence of a solution of (13) is ensured when

$$\text{p.v.} \int_{\mathbb{R}} \frac{G'_\epsilon(v)}{v} dv = 2\int_0^\infty \frac{G'_\epsilon(v)}{v} dv = 2\int_0^\infty \frac{G_\epsilon(v)}{v^2} dv = \frac{2}{\epsilon^2} \int_0^\infty \frac{G(v)}{v^2} dv > 1\,.$$

3. **Consequence of the modal analysis for the linearized problem: Stability of the single bump profile.** In the presence of unstable modes (which are frequency homogenous $\omega(k) = \omega^* k$) the solution of the V−D−B equation with initial data

$$\int_{\mathbb{R}} e^{ikx} \frac{G'(v)}{v-\omega^*} \hat{\rho}(k) dk,$$

has to be given by

$$f(t,x,v) = \int_{\mathbb{R}} e^{ikx} e^{-i\omega^* kt} \frac{G'(v)}{v-\omega^*} \hat{\rho}(k) dk,$$

and even for initial data in $\mathcal{S}(\mathbb{R})$ for $t > 0$ it is not defined (even in $\mathcal{S}'(\mathbb{R})$) unless $|\rho(k)| \le Ce^{-a|k|}$. In this case (which corresponds to analytic initial data [34, 42]) it exists up to a finite time $T^* = a/|\operatorname{Im}\omega^*|$ and may not exist for later time.

On the other hand a profile $v \mapsto G(v)$ with only one maximum leads to a stability result, robust with respect to the potentials and profiles $G(v)$. To precise the stability result we start with a the derivation of a formal energy identity for smooth functions and then return to the theory of strongly continuous groups of operators.

Proposition 2. *Assume that the profile $G(v)$ has only one bump or more precisely that:*

$$G'(v) := -H(v)(v-a) \quad \textit{with} \quad H(v) > 0, \quad a \in \mathbb{R}, \tag{14}$$

then any smooth solution $f(t,x,v)$ of the linearized Vlasov equation with potential V:

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) - G'(v)\, \partial_x \int_{\mathbb{R}} V(x-y)\Big(\int_{\mathbb{R}} f(t,y,w)dw\Big)dy = 0\,, \tag{15}$$

satisfies the energy identity,

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbb{R}\times\mathbb{R}} H^{-1}(v)(f(t,x,v))^2 dxdv + \int_{\mathbb{R}\times\mathbb{R}} V(x-y)\rho_f(x,t)\rho_f(x,t)dxdy\Big) = 0\,. \tag{16}$$

Proof. Let us introduce the notation $f(t,x,v) = H(v)\tilde f(t,x,v)$, multiply the equation (15) by $\tilde f$ and integrate over the phase-space (x,v) to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbb{R}\times\mathbb{R}} H^{-1}(v)(f(t,x,v))^2 dxdv\Big) \\ &\qquad + \int_{\mathbb{R}\times\mathbb{R}} \partial_x V(x-y)\rho_f(t,y)\int_{\mathbb{R}} H(v)(v-a)\tilde f(t,x,v)dvdydx = 0\,. \end{aligned} \tag{17}$$

Then observe that one has

$$\begin{aligned} &a\int_{\mathbb{R}\times\mathbb{R}} \partial_x V(x-y)\rho_f(t,y)\int_{\mathbb{R}} f(t,x,v)dvdydx \\ &\qquad = a\int_{\mathbb{R}\times\mathbb{R}} \partial_x V(x-y)\rho_f(t,y)\rho_f(t,x)dydx = 0\,. \end{aligned}$$

Therefore (17) turns out to be

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbb{R}\times\mathbb{R}} H^{-1}(v)(f(t,x,v))^2 dxdv\Big) \\ &\qquad - \int_{\mathbb{R}\times\mathbb{R}} V(x-y)\rho_f(t,y)\partial_x \int_{\mathbb{R}} vH(v)\tilde f(t,x,v)dvdydx = 0\,. \end{aligned} \tag{18}$$

This last term can be treated by integration of (15) with respect to the velocity v as

$$\partial_t \rho_f(t,x) + \partial_x \int_{\mathbb{R}} vH(v)\tilde f(t,x,v)dv = 0\,. \tag{19}$$

Eventually, pluging (19) into (18) one obtains (16). $\square$

With the above computation in mind, we consider linearized Vlasov equations with a positive semi-definite profile V near a profile $G(v)$ which satisfies the relation (14) and introduce the Hilbert space $\mathcal{H}_V$ of functions $f : \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\|f\|_{\mathcal{H}_V} := \int_{\mathbb{R}\times\mathbb{R}} H^{-1}(v)(f(x,v))^2 dvdx + \int_{\mathbb{R}\times\mathbb{R}} V(x-y)\rho_f(x)\rho_f(y)dxdy < \infty,$$

with the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ defined by

$$\langle f,g\rangle_{\mathcal{H}_V} = \int_{\mathbb{R}\times\mathbb{R}} H^{-1}(v)f(x,v)g(x,v)dxdv \;+\; \int_{\mathbb{R}\times\mathbb{R}} V(x-y)\rho_f(x)\rho_g(y)dxdy.$$

Remark 2. 1. Since $G(v)$ is a positive density the weight

$$H^{-1}(v) = -\frac{(v-a)}{G'(v)},$$

is unbounded for $|v| \to \infty$. Therefore $\mathcal{H}_V$ is a subspace of functions of $L^2(\mathbb{R}_x \times \mathbb{R}_v)$ with convenient decay for $v \to \infty$.

2. Positive semi-definite potentials are potentials V such that

$$\int_{\mathbb{R}\times\mathbb{R}} V(x-y)g(x)g(y)dxdy \geq 0, \tag{20}$$

for all continuous functions g with compact support. By Bochner Theorem (cf. §13 of Chapter XI in [43]) such potential are the Fourier transform of positive measures

$$V(x) = \int_{\mathbb{R}} e^{ixk} d\nu(k),$$

with $k \mapsto \nu(k)$ a non decreasing right-continuous bounded function This class obviously contain the Dirac mass and in this case the scalar product on $\mathcal{H}_V$ becomes

$$\langle f, g\rangle_{\mathcal{H}_V} = \int_{\mathbb{R}\times\mathbb{R}} H^{-1}(v)f(x,v)g(x,v)dxdv \; + \; \int_{\mathbb{R}} \rho_f(x)\rho_g(x)dx.$$

Next we introduce the unbounded operator A, defined as the restriction to $\mathcal{H}_V$ of the integro-differential operator

$$f \mapsto v\partial_x f - G' \, \partial_x(V * \rho_f)\,,$$

which is defined by its domain, i.e.

$$D(A) = \{f \in \mathcal{H}_V \;\; \text{s.t.} \;\; Af = v\partial_x f - G' \, \partial_x(V*\rho_f) \in \mathcal{H}_V\}$$

and one has the following

Theorem 3.1. *For positive semi-definite potential V and a profile G which satisfies the relation (14) the unbounded operator A is anti-adjoint in $\mathcal{H}_V$ and therefore is the generator of a strongly continuous group of unitary operators in this space.*

Proof. First the operator A is defined in $\mathcal{H}_V$ as the closure of its restriction to the space of smooth functions $f \in \mathcal{D}(\mathbb{R}_x \times \mathbb{R}_v)$. Moreover since $\mathcal{D} \subset D(A) \subset \mathcal{H}_V \subset L^2$ with dense embedding, then A is a closed operator on $\mathcal{H}_V$ with domain $D(A)$ dense in $\mathcal{H}_V$. Secondly by explicit computations, inspired by the formulas (17) and (18), one shows that for any $f \in \mathcal{S}(\mathbb{R}_x \times \mathbb{R}_v)$

$$\langle Af, f\rangle_{\mathcal{H}_V} = 0. \tag{21}$$

Hence A is the closure of an anti-symmetric operator and for any λ with $\mathrm{Re}\lambda \neq 0$ and in particular for $\lambda \in \mathbb{R}$, using (21) one has

$$\|(\lambda I + A)f\|_{\mathcal{H}_V}\|f\|_{\mathcal{H}_V} \geq |\langle(\lambda I + A)f, f\rangle_{\mathcal{H}_V}| = |\lambda| \|f\|^2_{\mathcal{H}_V}.$$

Hence operator $(\lambda I + A)$ is an isomorphism from it domain to its image which turns out to be a closed subspace of $\mathcal{H}_V$. To prove that A is anti-adjoint, following Kato (cf. Theorem V-3.16 or Problem V-3.31 in [27]), it is enough to prove that for any $\lambda \in \mathbb{R}$, $\lambda \neq 0$ $(\lambda I + A)(D(A))$ is a dense subspace of $\mathcal{H}_V$ or that the map $(\lambda I + A)$ is surjective onto $\mathcal{H}_V$. To do so, for $g \in \mathcal{D}(\mathbb{R}_v \times \mathbb{R}_x)$, one prove the existence of a solution $f \in \mathcal{H}_V$ of the equation

$$\lambda f + Af = g, \tag{22}$$

by direct computation in Fourier space where the equation (22) is equivalent to the system

$$\left(1-\widehat{V}(k)\int_{\mathbb{R}}\frac{G'(v)}{\omega+v}dv\right)\hat{\rho}_f(k)=\int\frac{\hat{g}(k,v)}{\lambda+ikv}dv\,,\ \text{with}\ \omega=-i\frac{\lambda}{k},\tag{23}$$

and

$$\hat{f}(k,v)=\frac{\hat{g}(k,v)}{\lambda+ikv}+ik\widehat{V}(k)\frac{\hat{\rho}_f(k)G'(v)}{\lambda+ikv}.\tag{24}$$

Using the fact that $\widehat{V}(k)\geq 0$ and that $G(v)$ has a unique maxima (a such that $G'(a)=0$) and proceeding as in the example 1 of the subsection 2.1 one observes that

$$\mathrm{Re}\,\left(\widehat{V}(k)\int_{\mathbb{R}}\frac{G'(v)}{\omega+v}dv\right)<0,$$

and therefore from (23), one has

$$|\hat{\rho}_f(k)|\leq\left|\int_{\mathbb{R}}\frac{\hat{g}(k,v)}{\lambda+ikv}dv\right|.\tag{25}$$

Eventually, with $g\in\mathcal{D}(\mathbb{R}_x\times\mathbb{R}_v)$, the inequality (25) and the relation (24) where the formula $G'(v)=-H(v)(v-a)$, the boundness of $\widehat{V}$ (cf. point 2 of Remark 2) and Cauchy-Schwarz inequality are used one obtains that there exists a constant $C=C(|\mathrm{Re}\,\lambda|^{-1},|\mathrm{Im}\,\lambda|,G',\widehat{V})<\infty$, such that $\|f\|_{\mathcal{H}_V}\leq C\|g\|_{\mathcal{H}_V}$ and thus using equation (22) we finally deduce that f belongs to $D(A)\subset\mathcal{H}_V$. Now since $\mathcal{D}(\mathbb{R}_x\times\mathbb{R}_v)$ is dense in $\mathcal{H}_V$ on one side and $(A,D(A))$ is closed and densely defined on $\mathcal{H}_V$ on the other side, we get that for every $\lambda\in\mathbb{C}\backslash i\mathbb{R}$, $\lambda\in P(A)$ (where $P(A)$ is the resolvent set of A) the map $(\lambda I+A)$ is bijective from $D(A)$ onto $\mathcal{H}_V$, and the resolvent $(\lambda I+A)^{-1}$ is a bounded operator with

$$\|(\lambda I+A)^{-1}\|_{\mathcal{L}(\mathcal{H}_V)}\leq\frac{1}{|\mathrm{Re}\,\lambda|}.$$

The proof ends by using Hille-Yosida Theorems for generation of semigroups and groups (cf. e.g. Corollary 3.7 of Chapter II in [15]). ☐

Corollary 1. *With the hypothesis of the Theorem 3.1, A is the generator of a strongly continuous unitary group and the Cauchy problem*

$$\frac{df}{dt}+Af=0,\quad f(0,x,v)=f_0(x,v),\tag{26}$$

has for any initial data $f_0\in\mathcal{H}_V$ a unique "weak" solution $f=e^{-itA}f_0\in\mathscr{C}(\mathbb{R}_t;\mathcal{H}_V)$. Moreover whenever $f_0\in D(A)$ this solution is strong (i.e. $f\in\mathscr{C}^1(\mathbb{R}_t;\mathcal{H}_V)\cap\mathscr{C}^0(\mathbb{R}_t;\mathcal{D}(A))$).

This statement is a direct consequence of the Theorem 3.1 in the frame of strongly continuous semigroups and groups or in otherwords Hille-Yosida Theorems (cf. section 2 and 3 of §1 of Chapter IX in [27] or Proposition 6.2 and 6.4 in [15] or else Theorem 7.4 in [9]).

Remark 3. 1. Since $G(v)$ is a positive density the weight

$$H^{-1}(v)=-\frac{(v-a)}{G'(v)},$$

is unbounded for $|v|\to\infty$. Therefore the Corollary 1 concerns the evolution of solutions with convenient decay at infinity.

2. The Theorem 3.1 has been stated for any Vlasov equation with a positive semi-definite potential. In particular it applies to the case where V is the Dirac distribution.

Moreover let V_1 and V_2 two positive semi-definite potentials with $V_1 - V_2$ also positive semi-definite, denote by A_1 and A_2 the corresponding anti-adjoint operators (with the same one bump profile $G(v)$). Then $\mathcal{H}_{V_1} \subset \mathcal{H}_{V_2}$ and for any initial data $f_0 \in \mathcal{H}_{V_1}$, $e^{-tA_2} f_0$ belongs to $\mathscr{C}(\mathbb{R}_t; \mathcal{H}_{V_2})$. Moreover if V_2 converges (in a weak sense to V_1) the solution $e^{-tA_2} f_0$ will converge in a convenient weak sense (details are omitted) to $e^{-tA_1} f_0$.

4. **Consequence of the modal analysis for the nonlinear Vlasov-Dirac-Benney equation.** The above analysis of the linearized-V−D−B equation leads by perturbation methods to well adapted instability or stability results for the nonlinear-problem.

For a theorem concerning the instability we denote by $\dot{H}^m$ the space of functions $f \in L^\infty(\mathbb{R}_x, L^1(\mathbb{R}_v))$ with, for $1 \le l \le m$, derivatives $\partial_x^l f \in L^2(\mathbb{R}_x; L^1(\mathbb{R}_v))$ equipped with the corresponding norm.

Definition 4.1. We say that a Cauchy problem $f_0 \mapsto S(t) f_0$, defined by a nonlinear dynamics on the phase space $(x, v) \in \mathbb{R} \times \mathbb{R}$, is locally ($\dot{H}^m$-$\dot{H}^1$) well-posed if there is a constant c_m such that for any initial datum $f_0 \in \dot{H}^m$, there exist a time $T > 0$ and a unique solution

$$S(t) f_0 \in L^\infty(0, T; \dot{H}^1)$$

such that

$$\operatorname{ess}\sup_{t \in (0,T)} \| S(t) f_0 \|_{\dot{H}^1} \le c_m \| f_0 \|_{\dot{H}^m} .$$

Theorem 4.2. *For every $m \in \mathbf{N}^*$, the Cauchy problem for the dynamics $S(t)$ defined by the V–D–B equation is not locally ($\dot{H}^m$-$\dot{H}^1$) well-posed.*

Proof. The proof follows perturbation techniques as developed by [23] and coworkers. It proceeds par contradiction. Assume that the problem is locally ($\dot{H}^m$-$\dot{H}^1$) well-posed. Introduce a profile $G(v)$ which generates unstable modes $\omega^* k$ and consider, for the dynamics $S(t)$, initial data of the form

$$f_0^s = G(v) + s\phi_0(x, v) \quad \text{with} \quad \phi_0(x, v) = \int_{\mathbb{R}} e^{ikx} \frac{G'(v)}{v - \omega^*} \hat{\rho}(k) dk \,. \tag{27}$$

If the nonlinear problem is ($\dot{H}^m$-$\dot{H}^1$) well-posed for $0 < t < T$, on this interval the function

$$\partial_s (S(t) f_0^s)_{|s=0}$$

belongs to $L^\infty(0, T; \dot{H}^1)$ and is a solution of the linearized problem. Hence by explicit computation in Fourier space, it has to be given by the formula

$$\partial_s (S(t) f_0^s)_{|s=0}(t, x, v) = \int_{\mathbb{R}} e^{ikx} e^{-i\omega^* kt} \frac{G'(v)}{v - \omega^*} \hat{\rho}(k) dk.$$

If the problem would be well posed, for initial data given by (27) with

$$\forall m > 0, \quad \lim_{|k| \to \infty} |k|^m |\hat{\rho}(k)| = 0 \quad \text{and} \quad \forall a > 0, \quad \lim_{|k| \to \infty} |\hat{\rho}(k)| e^{a|k|} = \infty, \tag{28}$$

(i.e. which belong to $\cap_m \dot{H}^m$) the solution of the linearized problem would exist for some time T. But the formula (28) shows that this is not possible. Hence the contradiction □

The formula (28) also indicates that, for initial data satisfying the condition

$$|\hat{\rho}(k)| \leq Ce^{-a|k|}, \tag{29}$$

the linearized problem remains well posed for $|t| < a/|\omega^*|$ and therefore the nonlinear problem may be also locally in time well posed. With the Paley-Wiener Theorem (cf. §4 of Chapter 6 in [43], or Theorem 1 of Chapter 1 in [34] or else Theorem 97 of Chapter V in [42]) which relates decay properties at infinity of a function (or a distribution) with analyticity of its Fourier transform, the condition (29) implies that the function $\rho(x)$ can be extended as an analytic function $\rho(x+iy)$ in the strip $|y| < a$. Therefore it is only in the analytic setting that a general Cauchy theorem for the V−D−B equation can be valid and this is the object of the following

Theorem 4.3. *Jabin-Nouri (2011)* [25]*: For any (x, v) analytic function $f_0(x, v)$ with*

$$\forall \alpha,\, m\,,\, n \quad \sup_x |\partial_{x^m} \partial_{v^n} f_0(x, v)|(1 + |v|)^\alpha = C(m, n)o(|v|)$$

there exists, for a finite time T, an analytic solution of the Cauchy problem.

5. **Relations with fluid mechanics.** However as observed above the linearization near a simple bump profile leads to a very stable evolution equation and this motivates several stability theorems. Some of them can be obtained by connections with different equations of fluid mechanics.

5.0.1. *The mono-kinetic solution.* Direct computations show that a phase space density

$$f(t, x, v) = \rho(t, x)\delta(v - u(t, x))$$

is a distributional solution of the V−D−B equation (1) if and only if its moments

$$\rho(t, x) = \int_{\mathbb{R}} f(t, x, v)dv \quad \text{and} \quad \rho(t, x)u(t, x) = \int_{\mathbb{R}} vf(t, x, v)dv$$

are solutions of the system

$$\partial_t \rho + \partial_x(\rho u) = 0\,, \quad \partial_t(\rho u) + \partial_x \left(\rho u^2 + \frac{\rho^2}{2}\right) = 0. \tag{30}$$

For $(\rho, u) \in \mathbb{R}_+ \times \mathbb{R}$ the system is strictly hyperbolic therefore the existence of a local in time (near $(\tilde{\rho}_0+\alpha, u_0)$ with $\alpha > 0$ and $(\tilde{\rho}_0, u_0) \in H^2(\mathbb{R})$) of smooth solutions is ensured (cf. [14]) . Observe that this result is in full agreement with the stability of example 2 of the section 2.1.

5.0.2. *Multi-kinetic solutions.* These observations can be generalized to multi-kinetic solutions of the form

$$f(t, x, v) = \sum_{1 \leq n \leq N} \rho_n(t, x)\delta(v - u_n(t, x))$$

with (ρ_n, u_n) solutions of the system

$$\begin{aligned} &\partial_t \rho_n + \partial_x(\rho_n u_n) = 0, \\ &\partial_t(\rho_n u_n) + \partial_x \left(\rho_n u_n^2\right) + \rho_n \partial_x \Big(\sum_{1 \leq \ell \leq N} \rho_\ell \Big) = 0. \end{aligned}$$

However this system is not always hyperbolic and the Cauchy problem is not always locally in time well posed. In particular for $N = 2$ and $(\rho_1, \rho_2, u_1, u_2) = (1, 1, a, -a)$ direct computations show that the system is hyperbolic (hence the Cauchy problem is well posed) if and only if $a^2 > 2$. Once again this is in full agreement with the Example 3 of section 2.1.

6. **The one-bump continuous profile and the genuine Benney equation.** The very robust stability of linearized Cauchy problem, near a one-bump profile $G(v)$, indicates that similar local in time results should hold for the full nonlinear equation with initial data near a one-bump profile $G(v)$.

As long as $v \mapsto f(t,x,v)$ remains (for (t,x) given a.e.) a one-bump continuous profile, with maximum equal to 1 for simplicity, i.e.

$$\sup_{v\in\mathbb{R}} f(t,x,v) = 1, \quad (t,x) \text{ a.e.},$$

one defines a.e. in $(x,a) \in \mathbb{R}\times[0,1]$ the functions $v_\pm(t,x,a)$ by the formula

$$v_-(t,x,a) \le v_+(t,x,a), \quad f(t,x,v_\pm(t,x,a)) = a,$$

and recover the one-bump profile $f(t,x,v)$ according to the reconstruction formula

$$f(t,x,v) = \int_0^1 \mathrm{Y}(v_+(t,x,a)-v) - \mathrm{Y}(v_-(t,x,a)-v))da, \tag{30}$$

where Y denotes the Heaviside function.

Direct computation shows that in this situation f is a distributional solution of the V−D−B equation if and only if contours $v_\pm(t,x,a)$ are solutions of the system

$$\partial_t v_\pm + v_\pm \partial_x v_\pm + \partial_x \rho = 0, \quad \rho(t,x) = \int_0^1 (v_+(t,x,a) - v_-(t,x,a))da. \tag{31}$$

If we introduce the mean density and velocity of the fluid labelled by the tag "a", defined respectively by

$$\varrho(t,x,a) = v_+(t,x,a) - v_-(t,x,a), \quad u(t,x,a) = \frac{1}{2}(v_+(t,x,a) + v_-(t,x,a)), \tag{32}$$

this system (31) is equivalent to the fluid type system

$$\begin{aligned} &\partial_t \varrho(t,x,a) + \partial_x(\varrho(t,x,a)u(t,x,a)) = 0, \\ &\partial_t u(t,x,a) + \partial_x \left(\frac{1}{2}u^2(t,x,a) + \frac{1}{8}\varrho^2(t,x,a)\right) + \partial_x \int_0^1 \varrho(t,x,b)db = 0, \end{aligned} \tag{33}$$

which was derived by Benney [3] as a model for water-waves (This is our reason for the name Vlasov-Dirac-Benney). Without the integral term $\partial_x \int_0^1 \varrho(t,x,a)da$ the infinite dimensional system (33) would be an infinite system of isentropic Euler equations since all the fluids "a" are decoupled. For such systems two types of results are available.

1. In any space dimension with smooth initial data the local in time existence uniqueness and stability of a solution (see [14]).

2. In one-dimensional space variable the existence of a global in time weak entropic solution ([11, 12]).

The proof of 1 relies on the fact that the energy

$$\mathcal{E}(\varrho,u) = \frac{1}{2}\int_{\mathbb{R}}\int_0^1 \left(\varrho(t,x,a)u^2(t,x,a) + \frac{1}{12}\varrho^3(t,x,a)\right) dadx, \tag{34}$$

is for the isentropic equation (in the sense of Peter Lax) a convex entropy (therefore we use below the name energy-entropy!).

The existence of such entropy implies that the system is hyperbolic. Therefore in one-dimensional space variable it has "Riemann invariants" which are used for the proofs of 2.

In present case the energy-entropy:

$$\begin{aligned}\mathcal{E}(\varrho, u) =& \frac{1}{2}\int_{\mathbb{R}}\int_0^1 \left(\varrho(t,x,a)u^2(t,x,a) + \frac{1}{12}\varrho^3(t,x,a)\right) dadx \\ &+ \frac{1}{2}\int_{\mathbb{R}}\left(\int_0^1 \varrho(t,x,a)da\right)^2 dx,\end{aligned} \tag{35}$$

has an extra integral term which makes the analysis more complicated but does not prevent the generalization of the proofs of 1 or 2.

The second author ([4]) already gave a local time result for the Cauchy problem inspired by 2. Using previous works of Teshukov [39, 40, 41] he extended the notion of Riemann invariants as singular integral operators acting on contours $v_\pm$ which in fact form the continuous spectrum of the first-order transport operator (cf. [4] for details).

Below we describe a new proof for the system (31) inspired by the proofs of results of the type 1. This proof is slightly simpler and requires much less regularity (in particular no regularity with respect to the a variable).

The first observation is that, for a one-bump profile, as above, the conserved energy can be expressed in term of the variables $v_\pm$. With the relation $f(t,x, v_\pm(t,x,a)) = a$, one has

$$\begin{aligned}\int_{\mathbb{R}} \frac{|v|^2}{2} f(v)dv &= \frac{1}{6}\int_{\mathbb{R}} \frac{d|v|^3}{dv} f(v)dv \\ &= -\frac{1}{6}\int_{\mathbb{R}} |v|^3 \frac{df}{dv}dv = \frac{1}{6}\int_0^1 (v_+^3(a) - v_-^3(a))da\,.\end{aligned}$$

Hence for any one-bump profile, using the notation $\mathbf{V} = (v_-, v_+)^t$,

$$\begin{aligned}\eta(\mathbf{V}) = \int_{\mathbb{R}}\Bigg(&\frac{1}{6}\int_0^1 (v_+^3(t,x,a) - v_-^3(t,x,a))da \\ &+ \frac{1}{2}\left(\int_0^1 (v_+(t,x,a) - v_-(t,x,a))da\right)^2\Bigg)dx,\end{aligned} \tag{36}$$

is an energy-entropy. If $v^\pm \in L^2 \cap L^\infty(\mathbb{R}\times(0,1))$ we can show that the map $\eta : \Omega \mapsto \Gamma$ is Frechet-differentiable, where Ω (resp. Γ) is an open subset of the Banach space $\left(L^2 \cap L^\infty(\mathbb{R}\times(0,1))\right)^2$ (resp. $\mathbb{R}$). Since Frechet-differentiability implies Gâteaux-differentiability with the same derivatives, and to avoid to manipulate norms in the Frechet-differentiation, let us compute the Gâteaux derivative (i.e. a directional derivative) of the map η which corresponds to the functional derivative of physicists. Therefore we obtain for the first-order derivative

$$(D_{\mathbf{V}}\eta)\delta\mathbf{V}_1 \quad = \quad \lim_{\epsilon_1\to 0}\frac{\eta(\mathbf{V}+\epsilon_1\delta\mathbf{V}_1) - \eta(\mathbf{V})}{\epsilon_1} \quad = \quad \left[\frac{d}{d\epsilon_1}\eta(\mathbf{V}+\epsilon_1\delta\mathbf{V}_1)\right]_{|\epsilon_1=0}$$

$$= \int_{\mathbb{R}} dx \int_0^1 da \begin{pmatrix} -\frac{v_-^2}{2} - (v_+ - v_-)\int_0^1 da \\ \frac{v_+^2}{2} - (v_+ - v_-)\int_0^1 da \end{pmatrix} \cdot \begin{pmatrix} \delta v_{1-} \\ \delta v_{1+} \end{pmatrix}$$

$$= \int_{\mathbb{R}} dx \int_0^1 da \, \frac{\delta \eta}{\delta \mathbf{V}} \cdot \delta \mathbf{V}_1. \tag{37}$$

and for the second-order derivative

$$\begin{aligned} &(D^2_{\mathbf{V}}\eta)(\delta\mathbf{V}_1, \delta\mathbf{V}_2) \\ &\quad = \lim_{\epsilon_1,\epsilon_2\to 0} \frac{\eta(\mathbf{V}+\epsilon_1\delta\mathbf{V}_1+\epsilon_2\delta\mathbf{V}_2) - \eta(\mathbf{V}+\epsilon_1\delta\mathbf{V}_1) - \eta(\mathbf{V}+\epsilon_2\delta\mathbf{V}_2) - \eta(\mathbf{V})}{\epsilon_1\epsilon_2} \\ &\quad = \left[\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\eta(\mathbf{V}+\epsilon_1\delta\mathbf{V}_1+\epsilon_2\delta\mathbf{V}_2)\right]_{|\epsilon_1=\epsilon_2=0} = \left[\frac{d}{d\epsilon_2}D_{\mathbf{V}}\eta(\mathbf{V}+\epsilon_2\delta\mathbf{V}_2)\delta\mathbf{V}_1\right]_{|\epsilon_2=0} \\ &\quad = \int_{\mathbb{R}} dx \int_0^1 da \begin{pmatrix} \delta v_{2-} \\ \delta v_{2+} \end{pmatrix}^t \begin{pmatrix} -v_- + \int_0^1 da & -\int_0^1 da \\ -\int_0^1 da & v_+ + \int_0^1 da \end{pmatrix} \begin{pmatrix} \delta v_{1-} \\ \delta v_{1+} \end{pmatrix} \\ &\quad = \int_{\mathbb{R}} dx \int_0^1 da \, (\delta\mathbf{V}_2)^t \, \frac{\delta^2\eta(\mathbf{V})}{\delta\mathbf{V}\delta\mathbf{V}}\delta\mathbf{V}_1. \end{aligned} \tag{38}$$

Therefore, from (38), the matrix-integral operator

$$\Sigma(t,x,a) = \frac{\delta^2\eta(\mathbf{V})}{\delta\mathbf{V}\delta\mathbf{V}}(t,x,a) = \begin{pmatrix} -v_-(t,x,a) + \int_0^1 da & -\int_0^1 da \\ -\int_0^1 da & v_+(t,x,a) + \int_0^1 da \end{pmatrix}, \tag{39}$$

should be a symmetrizer for the system (31) in the space $L^2(\mathbb{R}; L^2(0,1))$. Indeed this leads to the

Proposition 3. A priori estimate. *Any smooth solution $\mathbf{V} = (v_-, v_+)^t$ of the equation (31), satisfies the a priori nonlinear Gronwall estimate*

$$\begin{aligned} &\frac{d}{dt}\left(\|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \int_{\mathbb{R}\times(0,1)} (\Sigma(\mathbf{V})\partial_x^3\mathbf{V}, \partial_x^3\mathbf{V}) da dx\right) \\ &\qquad \le C\left(1 + \|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x^3\mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))}\right)^2. \end{aligned} \tag{40}$$

Proof. The fact that

$$\forall\, k \ge 1, \quad |\partial_x^k \rho(t,x)|^2 \le 2\int_0^1 |\partial_x^k \mathbf{V}(t,a,x)|^2 da,$$

is systematically used. C denotes different constants, all of them being independent of the solution and changing from line to line. In some of the formulas the variables (t,x,a) may be omitted. First observe that one has

$$\begin{aligned} \|\partial_x^2\rho\|^2_{L^\infty(\mathbb{R})} &\le C\left(\|\partial_x^3\rho\|^2_{L^2(\mathbb{R})} + \|\rho\|^2_{L^\infty(\mathbb{R})}\right) \\ &\le C\left(\|\partial_x^3\mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))} + \|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}\right). \end{aligned} \tag{41}$$

Then from the equation (31), using the maximum principle (with integration along characteristic curves), Young inequality and (41), one deduces the estimate

$$\begin{aligned}\partial_t\|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} &\le C\left(\|\partial_x\rho\|^2_{L^\infty(\mathbb{R})}+\|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}\right)\\ &\le C\left(\|\partial_x^3\mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))}+\|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}\right)\\ &\le C\left(1+\|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}+\|\partial_x\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}+\|\partial_x^3\mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))}\right)^2 .\end{aligned}\tag{42}$$

By differentiating with respect to the x variable the equation (31) we get

$$\partial_t\partial_x v_\pm + v_\pm\partial_x(\partial_x v_\pm) = -(\partial_x v_\pm)^2-\partial_x^2\rho,$$

which, using the maximum principle, Young inequality and (41), gives the estimate

$$\begin{aligned}&\partial_t\|\partial_x\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}\\ &\le C\left(\|\partial_x^2\rho\|^2_{L^\infty(\mathbb{R})}+\|\partial_x\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}+\|\partial_x\mathbf{V}\|^3_{L^\infty(\mathbb{R}\times(0,1))}\right)\\ &\le C\Big(\|\partial_x^3\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}+\|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}\\ &\quad+\|\partial_x\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}+\|\partial_x\mathbf{V}\|^3_{L^\infty(\mathbb{R}\times(0,1))}\Big)\\ &\le C\left(1+\|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}+\|\partial_x\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))}+\|\partial_x^3\mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))}\right)^2 .\end{aligned}\tag{43}$$

The next step involves the symmetrization of the equation (31) written in the form

$$\partial_t\mathbf{V}+M\partial_x\mathbf{V}=0,\tag{44}$$

with the matrix-integral operator M given by the formula

$$M(t,x,a)=\begin{pmatrix} v_-(t,x,a)-\displaystyle\int_0^1 da & \displaystyle\int_0^1 da\\ -\displaystyle\int_0^1 da & v_+(t,x,a)+\displaystyle\int_0^1 da\end{pmatrix}.\tag{45}$$

For later use observe that the derivatives of M have the following simple form

$$\forall\, k\ge 1,\quad \partial_x^k M(t,x,a)=\begin{pmatrix}\partial_x^k v_-(t,x,a) & 0\\ 0 & \partial_x^k v_+(t,x,a)\end{pmatrix}.$$

Then for the third-order x-derivative of $\mathbf{V}$ one has

$$\partial_t\partial_x^3\mathbf{V}+M\partial_x(\partial_x^3\mathbf{V})=R,\tag{46}$$

with

$$R=-\partial_x^3M\partial_x\mathbf{V}-3\partial_x^2M\partial_x^2\mathbf{V}-3\partial_xM\partial_x^3\mathbf{V},$$

and notice that, with the Gagliardo-Nirenberg interpolation inequality

$$\forall f\in H^2(\mathbb{R}),\qquad \|\partial_x f\|_{L^4}\le C\|\partial_x^2 f\|^{1/2}_{L^2}\|f\|^{1/2}_{L^\infty},$$

to bound the term $\|\partial_x^2M\partial_x^2\mathbf{V}\|_{L^2(\mathbb{R}\times(0,1))}$ one has

$$\|R\|_{L^2(\mathbb{R}\times(0,1))}\le C\|\partial_x\mathbf{V}\|_{L^\infty(\mathbb{R}\times(0,1))}\|\partial_x^3\mathbf{V}\|_{L^2(\mathbb{R}\times(0,1))}.\tag{47}$$

Next apply to the equation (46) the operator (39) and observe that for almost $t\ge 0$, $K=\Sigma M\in\mathcal{L}\left((L^2(\mathbb{R}\times(0,1)))^2\right)$ is a matrix-integral symmetric operator given by the formula

$$K = \begin{pmatrix} -v_-^2 \ + \displaystyle\int_0^1 da \cdot v_- \ + \ v_- \cdot \int_0^1 da & -v_- \cdot \displaystyle\int_0^1 da \ - \ \int_0^1 da \cdot v_+ \\ -v_+ \cdot \displaystyle\int_0^1 da \ - \ \int_0^1 da \cdot v_- & v_+^2 \ + \displaystyle\int_0^1 da \cdot v_+ \ + \ v_+ \cdot \int_0^1 da \end{pmatrix}.$$

For later use observe that the t-derivative of Σ is given by

$$\partial_t \Sigma = \begin{pmatrix} -\partial_t v_- + & 0 \\ 0 & \partial_t v_+ \end{pmatrix} = \begin{pmatrix} v_- \partial_x v_+ + \partial_x \rho & 0 \\ 0 & -v_+ \partial_x v_+ - \partial_x \rho \end{pmatrix}, \tag{48}$$

while the entries of the matrix $\partial_x K$ are given by

$$\begin{aligned} (\partial_x K)_{11} &= -2v_- \partial_x v_- + \int_0^1 da \cdot \partial_x v_- + \partial_x v_- \cdot \int_0^1 da\,, \\ (\partial_x K)_{12} &= (\partial_x K)_{21}^\star = -\partial_x v_- \cdot \int_0^1 da - \int_0^1 da \cdot \partial_x v_+\,, \\ (\partial_x K)_{22} &= 2v_+ \partial_x v_+ + \int_0^1 da \cdot \partial_x v_+ + \partial_x v_+ \cdot \int_0^1 da\,. \end{aligned} \tag{49}$$

where $T^\star$ designates the adjoint operator of T, with respect to the $L^2(\mathbb{R} \times (0,1))$-scalar-product. Therefore applying Σ to the equation (46), and taking the $L^2(\mathbb{R} \times (0,1))$-scalar-product against $\partial_x^3 \mathbf{V}$, leads to

$$\frac{d}{dt}(\Sigma \partial_x^3 \mathbf{V}, \partial_x^3 \mathbf{V}) = 2(\Sigma R, \partial_x^3 \mathbf{V}) + (\partial_t \Sigma \partial_x^3 \mathbf{V}, \partial_x^3 \mathbf{V}) + (\partial_x K \partial_x^3 \mathbf{V}, \partial_x^3 \mathbf{V})\,. \tag{50}$$

The three terms of the right hand side of (50) are estimated as follows. For the first term, using (39) and (47) one has

$$\begin{aligned} \|(\Sigma R, \partial_x^3 \mathbf{V})\| &\le \|\Sigma\|_{L^\infty} \|R\|_{L^2(\mathbb{R}\times(0,1))} \|\partial_x^3 \mathbf{V}\|_{L^2(\mathbb{R}\times(0,1))} \\ &\le C(1 + \|\partial_x \mathbf{V}\|_{L^\infty(\mathbb{R}\times(0,1))}) \|R\|_{L^2(\mathbb{R}\times(0,1))} \|\partial_x^3 \mathbf{V}\|_{L^2(\mathbb{R}\times(0,1))} \\ &\le C\left(1 + \|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x \mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x^3 \mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))}\right)^2. \end{aligned}$$

For the second term, using (48) and (41), one gets

$$\begin{aligned} &|(\partial_t \Sigma \partial_x^3 \mathbf{V}, \partial_x^3 \mathbf{V})| \\ &\le C\left(\|\mathbf{V}\|_{L^\infty(\mathbb{R}\times(0,1))} \|\partial_x \mathbf{V}\|_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x \rho\|_{L^\infty(\mathbb{R})}\right) \|\partial_x^3 \mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))} \\ &\le C\left(1 + \|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x \mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x^3 \mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))}\right)^2. \end{aligned}$$

Finally for the third term, using (49) and the same token, one obtains

$$\begin{aligned} |(\partial_x K \partial_x^3 \mathbf{V}, \partial_x^3 \mathbf{V})| &\le \|\partial_x K\|_{L^\infty(\mathbb{R}\times(0,1))} \|\partial_x^3 \mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))} \\ &\le C\left(1 + \|\mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x \mathbf{V}\|^2_{L^\infty(\mathbb{R}\times(0,1))} + \|\partial_x^3 \mathbf{V}\|^2_{L^2(\mathbb{R}\times(0,1))}\right)^2. \end{aligned}$$

With the insertion of these three estimates in (50), and estimates (42)-(43) the proof of the proposition (3) is completed. □

This leads to the following theorem

Theorem 6.1. *Let us introduce the functionnal space*

$$\mathcal{B}(T^*) = \left\{ \mathbf{V} \in \mathscr{C}\left(0, T^*; L^\infty(\mathbb{R}_x \times (0,1))\right) \cap L^\infty\left(0, T^*; L^2((0,1); H^3(\mathbb{R}_x))\right) \right\} \tag{51}$$

and the open subset $\mathcal{O}(m,M,T^*)$ *of* $\mathcal{B}(T^*)$*, defined by*

$$\begin{aligned}&\mathcal{O}(m,M,T^*)\\&\qquad=\{\mathbf{V}\in\mathcal{B}(T^*)\quad \textit{with}\quad 0<m<-v_-(t,x,a),\ v_+(t,x,a)<M<\infty\}\,.\end{aligned}\tag{52}$$

with $\mathbf{V}(t,x,a)=(v_-(t,x,a),v_+(t,x,a))^t$*.*

1. Assume that the initial data $\mathbf{V}(0,x,a)=(v_-(0,x,a),v_+(0,x,a))^t$ *satisfy for some given* $m>0$*, and* $M>0$ *the estimate*

$$m<-v_-(0,x,a)<M \qquad \textit{and} \qquad m<v_+(0,x,a)<M,\tag{53}$$

and the regularity property

$$\|\partial_x^3\mathbf{V}(0)\|_{L^2(\mathbb{R}\times(0,1))}\le\kappa<\infty,\tag{54}$$

then there exists a time $T^*=T^*(m,M,R)$ *such that the corresponding Cauchy problem, for the system*

$$\partial_t v_\pm+\partial_x\left(\frac{v_\pm^2}{2}+\int_0^1(v_+(t,x,a)-v_-(t,x,a))da\right)=0,\tag{55}$$

has a unique solution $(\mathbf{V}=(v_-(t,x,a),v_+(t,x,a))^t)\in\mathcal{O}(m,M,T^*)$*.*

2. Moreover if $\mathbf{V}(0,x,a)$ *is the weak limit (for instance in* $L^\infty(\mathbb{R}\times(0,1))$ *weak-*$*$*) of a sequence of functions* $\mathbf{V}^N(t,x,a)$ *which satisfy uniformly with respect to* N *the estimates (53) and (54) the corresponding solutions (defined in* $\mathcal{O}(m,M,T^*)$ *with* T^* *independent of* N*) converge (for instance in* $\mathcal{B}(T^*)$ *weak-*$*$*) to a function* $\mathbf{V}(t,x,a)$ *which is the solution of the problem (55) with the corresponding initial data.*

Proof. The a priori estimate (40) is an adaptation to the present case (where M includes in its expression integral operators) of the classical estimates for hyperbolic systems with entropy. Then the remaining part of the proof follows the lines of this classical case (see [14] for example). The main detail in the difference appears in the Gronwall estimate deduced from the relation (40). It would be of the type

$$\frac{dY}{dt}\le CY^2,$$

with

$$\begin{aligned}Y(t)=&\|\mathbf{V}(t)\|^2_{L^\infty(\mathbb{R}\times(0,1))}+\|\partial_x\mathbf{V}(t)\|^2_{L^\infty(\mathbb{R}\times(0,1))}\\&+\int_{\mathbb{R}\times(0,1)}(\Sigma(t)\partial_x^3\mathbf{V}(t),\partial_x^3\mathbf{V}(t))dadx,\end{aligned}$$

if the right hand side of (40) could be bounded by CY^2. In $Y(t)$ appears the expression

$$\begin{aligned}&(\Sigma(t)\partial_x^3\mathbf{V}(t),\partial_x^3\mathbf{V}(t))\\&\qquad=\int_{\mathbb{R}}dx\int_0^1da\ \left(-v_-(t,x,a)(\partial_x^3v_-(t,x,a))^2+v_+(t,x,a)(\partial_x^3v_+(t,x,a))^2\right)\\&\qquad+\int_{\mathbb{R}}dx\left(\int_0^1(\partial_x^3v_+(t,x,a)-\partial_x^3v_-(t,x,a))da\right)^2,\end{aligned}\tag{56}$$

which is non negative provided $-v_-(t,x,a)>0$ and $v_+(t,x,a)>0$. Moreover for $Y(t)$ to be finite at $t=0$, the hypothesis $-v_-(x,a,0)$, $v_+(x,a,0)<M$ are required.

However the relations

$$m < m_1 < -v_-(t,x,a),\ v_+(t,x,a) < M_1 < M,$$

are "open properties" and with $\partial_t \mathbf{V}$ bounded in L^∞, they remain valid for a finite time. With these observations one can construct say by iteration and for T^* small enough (as in [14] and [32]) the solution $\mathbf{V} \in \mathcal{O}(m,M,T^*)$. Indeed we prove the existence of a unique solution $\mathbf{V} \in \mathcal{O}(m,M,T^*)$ by determining it as a fixed point of the map that carries $\mathbf{U} = (u_-, u_+)^t \in \mathcal{O}(m,M,T^*)$ to the solution $\mathbf{V} \in \mathcal{O}(m,M,T^*)$ of the linearized system

$$\partial_t \mathbf{V} + M(\mathbf{U})\partial_x \mathbf{V} = 0, \tag{57}$$

with $M(\mathbf{U})$ given by (45), where we substitute $u_\pm$ to $v_\pm$. The system (57) is still symmetrizable by $\Sigma(\mathbf{U})$ (cf. (39)). Now using the fact that $\mathbf{U} \in \mathcal{O}(m,M,T^*)$, the same kind of energy estimates that (40) derived in Proposition 3 and the maximum principle (by using integration along characteristic curves or using a priori estimates of some characteristic functions of the solution such as for example $\max\{v_- + m - \int_0^t \|\partial_x \rho(\tau)\|_{L^\infty(\mathbb{R})} d\tau,\ 0\}$) we can show that equation (57) admits a solution $\mathbf{V} \in \mathcal{O}(m,M,T^*)$ for T^* small enough. Moreover, considering the equation satisfied by the difference of two solutions $\mathbf{V}_1,\ \mathbf{V}_2 \in \mathcal{O}(m,M,T^*)$ of (57), induced by two states $\mathbf{U}_1,\ \mathbf{U}_2 \in \mathcal{O}(m,M,T^*)$, and using the same token (symmetrizer $\Sigma(\mathbf{U})$, energy estimates of type (40) and maximum principle) we can show that the map which carries $\mathbf{U}$ to $\mathbf{V}$ is a contraction in a suitable metric for $\mathcal{B}(T^*)$ or $\mathcal{O}(m,M,T^*)$; and thus that the unique fixed point of that map will be the desired solution $\mathbf{V} \in \mathcal{O}(m,M,T^*)$ of (44) with initial condition $\mathbf{V}(0)$.

The point 2 is a direct consequence of the fact that the x-regularity estimate is uniform with respect to N. This is enough to pass to the limit in the equations. □

Remark 4. 1. As noticed above the system, (31) is equivalent with the change of unknowns (32) to the Benney equation (33), hence the theorem 6.1 provides with this change of variables a similar treatment of the Cauchy problem for this system. In fact a direct proof (with slightly more complicated estimates) could be done for this system using the symmetrizer

$$\frac{\delta^2 \mathcal{E}(\rho,u)}{\delta \mathfrak{m} \delta \mathfrak{m}}(t,x,a) = \begin{pmatrix} \frac{1}{4}\varrho(t,x,a) + \displaystyle\int_0^1 da & u(t,x,a) \\ u(t,x,a) & \varrho(t,x,a) \end{pmatrix}, \tag{58}$$

with $\mathfrak{m} = (\varrho, u)^t$.

2. The essential "geometric hypothesis" are

$$-M < v_-(x,a,0) < -m < 0 < m < v_+(x,a,0) < M\,.$$

Besides this requirement the present proof uses no regularity or monotonicity of the map $a \mapsto v_\pm(t,x,a)$. However with the equation

$$\partial_t \partial_a (v_\pm) + \partial_x (v_\pm(\partial_a v_\pm)) = 0, \tag{59}$$

obtained by differentiating (55) with respect to the a variable, one observes that the solution $v_\pm(x,a,t)$ given by the Theorem 6.1 would preserve the monotonicity of the functions $a \mapsto v_\pm(x,t,a)$ up to the time T^* whenever such property holds for $t = 0$ [13].

Keeping in mind the point 2 of the above remark one can consider for $\mathbf{V}(t,x,a)$ piecewise constant functions with respect to the a variable as described in the following,

Proposition 4. *Assume that the functions $v_\pm(0,x,a)$ are piecewise constant and defined by the formula*

$$\text{for } 1 \le j \le N \quad \text{and} \quad \frac{N-j}{N} < a \le \frac{N-j+1}{N}, \quad v_\pm(0,x,a) = v_\pm(0,x,j),$$

with for $1 \le j \le N$,

$$v_\pm(0,x,j) \in H^3(\mathbb{R}) \quad \text{and} \quad m < -v_-(0,x,j),\, v_+(0,x,j) < M,$$

then the Cauchy problem associated to the waterbag equation (55) with the corresponding initial data $v_\pm(0,x,a)$ has a unique solution $\mathbf{V} \in \mathcal{O}(m,M,T^*)$.

This proposition is a direct consequence of the Theorem 6.1 because the initial data satisfy the hypothesis of this theorem.

To conclude this section the above results are applied to the original Vlasov-Dirac-Benney equation with the following

Theorem 6.2. *Assume that the functions $v_\pm(0,x,a)$ satisfy the hypothesis of the Theorem 6.1, that $a \mapsto v_-(0,x,a)$ is increasing and $a \mapsto v_+(0,x,a)$ is decreasing then the Vlasov-Dirac-Benney equation (1) with initial data given by*

$$f_0(x,v) = \int_0^1 (\mathrm{Y}(v_+(0,x,a)-v) - \mathrm{Y}(v_-(0,x,a)-v))da, \tag{60}$$

(with Y denoting the Heaviside function) has for $0 < t < T*$ a unique solution

$$f(t,x,v) \in L^\infty(0,T^*;L^p(\mathbb{R}_x \times \mathbb{R}_v)) \quad \text{for all} \quad 1 \le p \le \infty,$$

with

$$\rho(t,x) = \int_{\mathbb{R}} f(t,x,v)dv \in L^\infty(0,T^*;H^3(\mathbb{R})),$$

which is given by the formula

$$f(t,x,v) = \int_0^1 (\mathrm{Y}(v_+(t,x,a)-v) - \mathrm{Y}(v_-(t,x,a)-v))da\,. \tag{61}$$

Moreover when f_0 is defined as the limit for $N \to \infty$ of a sequence of functions

$$f_0^N(x,v) = \int_0^1 (\mathrm{Y}(v_+^N(0,x,a)-v) - \mathrm{Y}(v_-^N(0,x,a)-v))da,$$

where the $a \mapsto v_\pm^N(0,x,a)$ are monotonic functions ($a \mapsto v_-(0,x,a)$ non-decreasing and $a \mapsto v_+(0,x,a)$ non-increasing) satisfying, uniformly with respect to N, the hypothesis

$$-M < v_-^N(0,x,a) < -m, \quad 0 < m < v_+^N(0,x,a) < M,$$
$$\|v_\pm(0)\|_{L^2((0,1);H^3(\mathbb{R}_x))} < \infty,$$

the corresponding solutions exist on a time interval T^* independent of N and one has

$$f_N(t,x,v) \rightharpoonup f(t,x,v) \quad \text{in} \quad L^\infty(0,T^*;L^p(\mathbb{R}_x \times \mathbb{R}_v)) \text{ weak} - *,$$

$$\rho_N(t,x) = \int_{\mathbb{R}} f^N(t,x,v) \rightharpoonup \rho(t,x) = \int_R f(t,x,v)dv \text{ in } L^\infty(0,T^*;H^3(\mathbb{R})) \text{ weak} - *.$$

Proof. We first prove that the monotonicity of the functions $a \mapsto v_\pm(t,x,a)$ is preserved by the dynamics. Let us set $w_\pm = v_\pm(b) - v_\pm(a)$, $\tilde{v} = v_\pm(b) + v_\pm(a)$, and form the equation for the difference $w_\pm$ of two solutions of (55), which is equivalent to integrate the equation (59) with respect to the a variable between a and b. Therefore multiplying the resulting equation by the derivative of a convex regularization of the modulus of $w_\pm$, integrating with respect to the x variable and using the fact that $v_\pm(t) \in L^1((0,1); W^{1,1}(\mathbb{R}_x))$ (since $v_\pm(t) \in L^2((0,1); H^3(\mathbb{R}_x))$ and using Sobolev embeddings) we can show the property

$$\frac{d}{dt}\|w_\pm\|_{L^1(\mathbb{R}_x)} \leq 0. \tag{62}$$

Now using Crandall-Tartar result [13] about the relation between nonexpansive (i.e. (62)) and order preserving (i.e. monotonicity of the functions $a \mapsto v_\pm(t,x,a)$) mappings, we get, after time integration of (62), the desired result. Since now the monotonicity of the functions $a \mapsto v_\pm(t,x,a)$ is preserved by the dynamics, it implies that one can use the formula (61) to reconstruct the solution f or f^N. In particular observe that the f^N are solution of a Liouville equation

$$\partial_t f^N + v\partial_x f^N + \partial_x \rho^N \partial_v f^N = 0, \quad \rho^N(t,x) = \int_{\mathbb{R}} f^N(t,x,v)dv,$$

with ρ^N uniformly bounded in $L^\infty(0,T; H^3(\mathbb{R}))$ which can be used to consider the limit $N \to \infty$. □

Remark 5. The biggest constraint in the above construction is the fact that the functions $a \mapsto v_\pm(0,x,a)$ have to be defined on a fixed interval (say $a \in [0,1]$) and bounded above and below. This implies for the initial profiles $v \mapsto f_0(x,v)$ the following x independent properties.
(H1) There exist an x independent constant $0 < M < \infty$ such that

$$|v| \geq M \Rightarrow f_0(x,v) = 0. \tag{63}$$

(H2) There exist an x independent constant $0 < m < \infty$ constant such that

$$|v| \leq m \Rightarrow f_0(x,v) = 1. \tag{64}$$

(H3) The map $v \mapsto f_0(x,v)$ is non-decreasing on the interval $]-\infty, -m]$ and non-increasing on the interval $[m, +\infty[$. In short it is a "plateau" profile near $v = 0$.

The instability Theorem 4.2 implies that the one bum shape of the profile has to be preserved by the dynamics and therefore the hypothesis (H2) and (H3) seem almost optimal. On the other hand in the right hand side of (56) appears the $\partial_x^3 \mathbf{V}$ quadratic term

$$\int_{\mathbb{R}} dx \int_0^1 da \ \left(-v_-(t,x,a)(\partial_x^3 v_-(t,x,a))^2 + v_+(t,x,a)(\partial_x^3 v_+(t,x,a))^2\right)$$

where $(\partial_x^3 v_\pm(t,x,a))^2$ are multiplied by the factor $|v_\pm(t,x,a)|$. Therefore with estimates adapted to this factors (treated as weights) it may be possible to relax the hypothesis (H1) and consider one bump initial profiles with unbounded support?

On the other hand it is important to observe no other regularity with respect to v is needed and the introduction of the $v_\pm^N$ satisfying the hypothesis of the proposition shows the validity of the waterbag model (cf. [4] and [5] for details) as a convenient approximation for the continuous model.

Remark 6. Some different physical scalings lead, instead of V−D−B equation, to other variants and in particular to the constant density Vlasov equation, i.e.

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \quad \int_{\mathbb{R}^d} f(t,x,v)dv = 1,$$

(cf. [6, 7, 8, 22, 24]) where the electric field plays the role of the Lagrange multiplier of the constraint "constant density". For instance, in more than one dimension ($d >$

1) this equation has non trivial mono-kinetic solutions $f(t,x,v) = \rho_0\delta(v - u(t,x))$, where ρ_0 is constant and $u(t,x)$ is the solution of the incompressible Euler equation. For $d = 1$, Grenier [22] proves an instability theorem (Theorem 1.1) which is the counterpart of Example 4 of section 2 and of Theorem 4.2 of section 4. Moreover by energy methods he has proven, for one-bump profile a stablility result (Theorem 2.1) which share much in common with the present Theorem 6.1.

7. **The Vlasov-Dirac-Benney equation at the cross road of semi-classical limits, fluid mechanics and integrability.** We have shown that with the singularity of the potential (the Coulomb potential being replaced by a Dirac mass) dynamics may be very unstable and that this issue is closely related to the behavior of the linearized solution as a perturbation of different initial profiles. For instance perturbations of the "one-bump profile" are locally stable (both for the linearized and the nonlinear problem) while "bump in tail" profiles may lead to ill posed problems (also both for the linearized and the nonlinear V−D−B equation).

It has been also observed that stability results can be consequences of the relation of the Vlasov equation with equations in fluid mechanics. For the classical Vlasov equation this is not a new idea. Since the paper of Brenier [6] this point of view appeared to be very fruitful for the analysis of singular limits cf. for instance Han-Kwan [24], Loeper [31] and the contributions [4] and [5] of the second author. However for what we dubbed V−D−B the connection was already formally made by Zakharov in 1980 [44]. Moreover he observed formal relations between the Vlasov equation, and the WKB or semi-classical limits of the Non-Linear Schrödinger equation.

Therefore we would like to emphasize that such formal semi-classical limits turn out to be "rigorously proven limits" only in cases which also correspond to the stability near one-bump profile.

We start from the Schrödinger equation in $\mathbb{R}^d$ with a time-dependent potential $V(t,x)$

$$i\hbar\partial_t\psi = \mathcal{H}(\hbar, V(t))\psi = -\frac{\hbar^2}{2}\Delta\psi + V(t,x)\psi, \tag{65}$$

which defines a unitary dynamics in $L^2(\mathbb{R}^d)$ and therefore we assume that the wave-function ψ satisfies the normalization condition,

$$\int_{\mathbb{R}^d} |\psi(t,x)|^2 dx = 1\,.$$

With

$$V(t,x) = \int_{\mathbb{R}^d} \mathcal{V}(x-y)|\psi(t,y)|^2 dy, \tag{66}$$

one has the family of self consistent Schrödinger equations with in particular the Schrödinger Poisson equation when $\mathcal{V}$ is the Coulomb Potential or the Non-Linear Schrödinger equation when $\mathcal{V}$ is the Dirac mass.

On the other hand with the introduction of the commutator $[A, B] = AB - BA$, the so-called self-consistent Von Neumann equation

$$i\hbar\partial_t K_\hbar(t) = [K_\hbar(t), \mathcal{H}(\hbar, V(t))] \quad \text{with} \quad V(t,x) = \int_{\mathbb{R}^d} \mathcal{V}(x-y)K_\hbar(t,y,y)dy, \tag{67}$$

defines a dynamics on trace 1 self-adjoint unitary operators in $L^2(\mathbb{R}^d)$ (with kernel denoted by $K_\hbar(t,x,y)$). In particular whenever $\psi_\hbar(t)$ is solution of the equation (65) with $V(t)$ given by (66), $K_\hbar(t,x,y) = \psi_\hbar(t,x) \otimes \overline{\psi_\hbar(t,y)}$ is a solution of the

Von-Neumann equation (67). Eventually we introduce the Wigner transform of the operator $K_\hbar(t)$

$$W_\hbar(t,x,v)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-iy\cdot v}K_\hbar\Big(t,x+\frac{\hbar}{2}y,x-\frac{\hbar}{2}y\Big)dy, \tag{68}$$

and we observe that its formal $\hbar\to 0$ limit $W(t,x,v)$ is a solution of the Vlasov equation

$$\begin{aligned}&\partial_t W(t,x,v)+v\cdot\nabla_x W(t,x,v)\\&\qquad-\nabla_x\left(\int_{\mathbb{R}^d}\mathcal{V}(x-y)\int_{\mathbb{R}^d}W(t,y,w)dwdy\right)\cdot\nabla_v W(t,x,v)=0,\end{aligned}$$

with

$$W_0(x,v):=W(0,x,v)=\lim_{\hbar\to 0}W_\hbar(0,x,v). \tag{69}$$

Such results are proven when the potential $\mathcal{V}$ is smooth enough (cf. [30] or [19]).

For the Non-Linear Schrödinger equation and for its formal limit the V−D−B equation the situation is completely different. Since the Cauchy problem may be ill posed (cf. Theorem 4.2) there are in general no chances of such convergence (even for $\mathscr{C}^\infty$ data and small time). However, for $d=1$, convergence should hold for initial data of the form

$$K_\hbar(0,x,y)=\int_{\mathbb{R}}e^{i\frac{x-y}{\hbar}v}W_0\left(\frac{x+y}{2},v\right)dv,$$

with $W_0(x,v)$ being a one-bump profile satisfying in term of $v_\pm(0,x,a)$ the hypothesis of the Theorem 6.1 because the limit problem is well posed. We are not aware of such result. On the other hand if W_0 (cf. (69)), is analytic (satisfying the Jabin-Nouri hypothesis [25]) convergence should hold for a finite time. Here also, to the best of our knowledge there is no general proof of this fact. However in the WKB limit there is a contribution of P. Gerard [18] which may be generalized. This WKB limit refers to the $\hbar$ scaling of the equation (as above) and to initial data in the form

$$\psi_h(0,x)=\sum_{1\le k\le N}\rho_k(x)e^{i\frac{S_k(x)}{\hbar}}$$

which give for the Wigner transform at time $t=0$,

$$W_\hbar(0,x,v)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-iyv}\psi_\hbar\Big(0,x+\frac{\hbar}{2}y\Big)\overline{\psi}_\hbar\Big(0,x-\frac{\hbar}{2}y\Big)dy. \tag{70}$$

Now under general hypothesis ($\rho_k\in L^1_{\text{loc}},\ \partial_x S_k\in W^{1,1}_{\text{loc}}$) and the $\partial_x S_k(x)$ linearly independent one has

$$W_\hbar(0,x,v)\to\sum_{1\le k\le N}\rho_k(x)\delta(v-\nabla S_k(x)),\quad\text{in }\ \mathcal{D}'(\mathbb{R}). \tag{71}$$

For $N=1$, this corresponds to a mono-kinetic initial data. This is a case (also the “extreme case” of the one-bump profile) of initial data which give for V−D−B equation a local in time (before the appearance of singularities) stable solution. Therefore in this setting one can expect that the Wigner transform of $\psi_\hbar(t,x)\otimes\overline{\psi_\hbar(t,y)}$ will converge to the solution of V−D−B equation. And in fact several proofs of such convergence are available (as said above [18] in the analytic case) but also Grenier [20, 21] with a proof based on a modification of the Madelung transform and finally Jin, Levermore and McLaughlin [26].

Note also that the case $N > 1$, in formula (70), has been considered by Zakharov [44] with formal proofs of convergence. These proofs should completely work in the analytic case as an application of [18]. In less regular cases for example with $N = 2$ and (71) the examples of section 5.0.2 lead to the conjecture that with non analytic initial data, local in time convergence may hold in some cases but not in every cases.

The proof in ([26]) which holds for the mono-kinetic limit is based on the complete integrability of the Non-Linear Schrödinger equation by inverse scattering. And therefore the V−D−B equation appears to share many properties of integrable systems. Zakharov [44] says “it is integrable in a certain sense”! And he insists on the existence, for the genuine Benney equation of an infinite number of integrals of motion obviously related to the infinite set of invariants for the nonlinear Schrödinger equation and to the infinite set of conserved quantities (entropy) for the 2×2-equations of fluid mechanics.

8. **Conclusion.**

This contribution has been essentially devoted to a singular one-dimensional version of the Vlasov equation where the potential, which in the original case is derived from the Laplace equation, has been replaced by the Dirac mass.

Having briefly recalled that there are good physical reasons to consider this equation we have emphasize the mathematical properties of this problem.

The spectral analysis share much in common with the approach of Penrose (because of the one-dimensional structure and of the fact that the potential is positive semi-definite). However due to the singularity the effect of the initial data on the behavior of the solution (both for the linearized problem and for the original nonlinear equation) are much more drastic than for the classical Vlasov–Poisson equation. The case where the problem is locally in time well posed are treated thanks to interpretation in term of fluid mechanic. This justifies the name “Benney” in our contribution.

As said above the stability results are very sensitive to the geometrical structure of the initial data and this sensitivity persists all over the article from the property of the linearized problem to the stability analysis of the nonlinear one and to its interpretation as a WKB limit of nonlinear Schrödinger type equations.

What we dubbed energy-entropy quantities are in fact invariants of the dynamics and could be related at every level of the analysis to an intrinsic hamiltonian structure of the problem and should play the same role as the Casimir in the stability theory of Arnold for the two-dimensional Euler equation (cf. for instance [1]).

Therefore this leaves some room for further studies both on one hand for application to approximation and numerical analysis and on the other hand, at intrinsic mathematical theory, for the role that this type of equation may play at the cross road of different limits that share in common some hamiltonian structures. Eventually one may consider perturbation of profiles $G(v)$ with “multi-bumps” but small enough so that there would be no unstable mode:

$$\forall \omega^* \in \Im_+, \qquad \left| 1 - \int_{\mathbb{R}_v} \frac{G'_\epsilon(v)}{v - \omega^*} dv \right| > \eta. \tag{72}$$

In this case one can describe the evolution of the linearized problem (near this profile) by a distribution groups of operators [29, 10] as described in [2]. This means a “almost well-posed Cauchy problem” or more precisely an evolution equation well posed with a finite lost of regularity. In this setting it may be possible that the nonlinear problem be approached with the Nash Moser Theorem. Such a construction seems to require not only more regularity with respect to the x variable but also to the v. And this type of hypothesis are definitely not satisfied for approximations like the multi water-bags considered above.

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