Hamiltonian Structure, Fluid Representation and Stability for the Vlasov–Dirac–Benney Equation

Claude Bardos and Nicolas Besse

Pour Walter Craig, en remerciement pour ses contributions scientifiques et son amitié.

Abstract This contribution is an element of a research program devoted to the analysis of a variant of the Vlasov–Poisson equation that we dubbed the Vlasov–Dirac–Benney equation or in short V–D–B equation. As such it contains both new results and efforts to synthesize previous observations. One of main links between the different issues is the use of the energy of the system. In some cases such energy becomes a convex functional and allows to extend to the present problem the methods used in the study of conservation laws. Such use of the energy is closely related to the Hamiltonian structure of the problem. Hence it is a pleasure to present this article to Walter Craig in recognition to the pioneering work he made for our community, among other things, on the relations between Hamiltonian systems and Partial Differential Equations.

1 Introduction

This article extends a program (cf. [1, 2]) devoted to the mathematical analysis of an avatar of the Vlasov–Poisson equation, where the "Coulomb potential" is replaced by the Dirac mass. Since it was proposed by Benney [3] and Zakharov [28] for the description of water waves, it is dubbed Vlasov–Dirac–Benney equation (or in short V–D–B equation). Therefore the V–D–B equation reads

C. Bardos (✉)
Laboratoire Jacques-Louis Lions, Université Denis Diderot, Paris, France
e-mail: claude.bardos@gmail.com

N. Besse
Département Physique de la Matière et des Matériaux, Institut Jean Lamour UMR CNRS 7198, Université de Lorraine, BP 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France
e-mail: nicolas.besse@univ-lorraine.fr

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$$\partial_t f + v \cdot \nabla_x f - \nabla_x \rho_f \cdot \nabla_v f = 0\,, \quad \text{with} \quad \rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)dv. \tag{1}$$

And the classical conservation of mass and energy turn out to be given by the formula,

$$\partial_t \int_{\mathbb{R}^d_v} f(t,x,v)dv + \nabla_x \cdot \int_{\mathbb{R}^d_v} vf(t,x,v)dv = 0,$$

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} \frac{|v|^2}{2} f(t,x,v)dxdv &= -\int_{\mathbb{R}^d_x} \nabla_x \rho_f(t,x) \cdot \int_{\mathbb{R}^d_v} \frac{|v|^2}{2} \nabla_v f(t,x,v)dxdv \\
&= \int_{\mathbb{R}^d_x} \rho_f(t,x) \nabla_x \cdot \int_{\mathbb{R}^d_v} vf(t,x,v)dxdv \\
&= -\int_{\mathbb{R}^d_x} \rho_f(t,x)\partial_t \rho_f(t,x)dx,
\end{aligned}$$

or eventually

$$\frac{d\mathscr{E}}{dt} = \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^d_x} dx \left(\int_{\mathbb{R}^d_v} dv\, \frac{|v|^2}{2} f(t,x,v) + (\rho_f(t,x))^2\right)\right) = 0.$$

1.1 Some Physical Motivations for the Introduction of the Dirac Potential

One of the many physical motivations for the introduction of this equation is the description of a plasma constituted of ions in a background of "adiabatic" electrons which instantaneously reach a thermodynamical equilibrium (i.e. electrons follow a Maxwell–Boltzmann distribution). Therefore the charge density of electrons is given in term of the electrical potential Φ_ϵ by the formula

$$\rho_- = \rho_0 e^{-\frac{e\Phi_\epsilon}{k_B T_e}},$$

with k_B the Boltzmann constant, e the electron charge and T_e the equilibrium temperature of electrons. Finally the parameter ϵ represents the Debye length. Hence the "Coulomb law" couples the electrical potential Φ_ϵ to the charge density such that

$$-\epsilon^2 \Delta\Phi_\epsilon = \rho_\epsilon - \rho_0 e^{-\frac{e\Phi_\epsilon}{k_B T_e}}\,, \quad \text{with} \quad \rho_\epsilon = \int_{\mathbb{R}^d_v} f_\epsilon(t,x,v)dv.$$

Now since the electrical potential energy $e\Phi_\epsilon$ is supposed to be small in comparison to the kinetic energy $k_B T_e$, i.e $|e\Phi_\epsilon/(k_B T_e)| \ll 1$, after linearization on the exponential function, at first order we get

$$-\epsilon^2 \Delta \Phi_\epsilon = \rho_\epsilon - \rho_0 + \frac{e\rho_0}{k_B T_e} \Phi_\epsilon .$$

Setting ϵ to zero (quasineutrality assumption) and since ρ_0 and T_e are supposed to be constant, we obtain for the electric field E_ϵ the expression

$$E_\epsilon = -\nabla_x \Phi_\epsilon = \frac{k_B T_e}{e\rho_0} \nabla_x \int_{\mathbb{R}_v^d} f_\epsilon(t, x, v) dv,$$

which appears in the Vlasov equation (1).

1.2 Some Mathematical Motivations for this Analysis

Since in the Eq. (1) the electric field E is given in term of the electrons density by an operator of order 1, while in the classical Vlasov–Poisson case it is given by an operator of degree -1, the solution is much more dependent on the initial data. Therefore, while for the classical Vlasov–Poisson equation the issue is the large time asymptotic behavior, here what is at stake is the well-posedness of the problem in term of the initial data. On the other hand since the electrical potential is given by a purely local operator there exists a strong connection between the dynamics of hyperbolic systems of conservation laws and the V–D–B equation. This connection appears even more clearly when one uses for the Vlasov equation a kinetic representation of the form (cf. Sect. 3.2)

$$f(t, x, v) = \int_M \rho(t, x, \sigma) \delta(v - u(t, x, \sigma)) d\sigma, \tag{2}$$

which leads to non local "operator type" conservation laws.

For such conservation laws the invariants play an essential role and as expected, they coincide (cf. Theorem 6) with the Lax–Godunov conserved quantities. When such invariants turn out to be convex (with respect to the parameters of the dynamics) they play the role of convex entropies and ensure the local-in-time stability and well-posedness of the Cauchy problem.

As this is the case for the most general Vlasov equations (as explained for instance in [22]) the present V–D–B equation can be viewed as a Hamiltonian system related to the minimization of an energy. Moreover the same point of view can be used to formalize the relations between classical and quantum mechanics via semi-classical (WKB) limits and Wigner measure (cf. Sect. 6). Such convergence will be always true at the formal level, or with analytical initial data. However, as expected, proofs in the Distributions (or Sobolev) setting will be available only when the limit enjoys the same stability i.e. mostly in the case where a convex entropy is present. Even if the analyticity hypothesis is not "physical", conclusions that follow are important, and especially in the case of the one-dimensional space variable. Then the cubic nonlinear Schrödinger equation and its generalization as infinite systems of coupled nonlinear Schrödinger equations (cf. [28]) are integrable systems with a rich algebraic structure including in particular the construction of infinite family of conservation laws. In the semi-classical limits these structures (at least for analytic initial data) do persist and make the one-dimensional-space-variable V–D–B equation a quasi-integrable system in the sense of [28].

The paper is then organized as follows. First the emphasis is put on the one-dimensional space variable which as quoted above contains more mathematical structure and provides also more explicit examples. To underline the dimension one in the corresponding equations, the symbol ∇_x and ∇_v are replaced by the symbol ∂_x and ∂_v. In Sect. 2, the analysis of the linearized problems turns out to be (and this should not be a surprise) in full agreement with the properties of the fully nonlinear systems. Moreover this produces also a natural tool for the study of nonlinear perturbations which is the object of the next section.

In the Sect. 3, the Hamiltonian structure and the fluid representation of the kinetic V–D–B equation are described. In this setting, under strong analyticity hypothesis a local-in-time stability result can be proven and this is the object of the Theorem 5. To obtain stability results with finite order regularity, the entropies have to be introduced and compared with the classical invariants of the Hamiltonian system. This is the object of the Sect. 4 and Theorem 6. The next Sect. 5 is devoted to several examples of application.

For the discussion of the semi-classical limits in the Sect. 6, we follow similar route. First formal computations are given. Then there are validated with analyticity hypothesis (cf. Theorem 9). Such results are compared with a theorem of Grenier which is valid in any space variable, with Sobolev type regularity hypothesis, but which concerns only the Wigner limit of "pure states" i.e. mono-kinetic solutions of the V–D–B equation.

As a conclusion we return to the relation between Wigner limit and inverse scattering.

2 Properties of the Linearized Problem and Consequences

Long time ago, it has been observed that x-independent solutions

$$v \mapsto G(v) \geq 0 \quad \text{with} \quad \int_{\mathbb{R}} G(v)dv = 1,$$

are stationary solutions of the Vlasov–Poisson equation. Same simple observation is also valid for the V–D–B equation. Writing

$$f(t,x,v) = G(v) + \tilde{f}(t,x,v),$$

retaining only the linear terms in $\tilde{f}$ and omitting henceforth the tilde notation, one obtains the evolution equation

$$\partial_t f + v\nabla_x f - G'(v)\partial_x \rho_f(t,x) = 0.$$

It then turns out that in one space dimension the spectral analysis, hence the stability, can be described in term of the shape of the stationary profile $v \mapsto G(v)$. In particular:

i) One can prove for the classical Vlasov–Poisson equation (this goes back to Kruskal [19]) that if the profile has only one maximum (one bump profile) then the solution is described in a convenient Hilbert space by a unitary group and therefore is stable. This remark can be adapted to the V–D–B equation and is shortly described below.
ii) In the presence of several extrema, a criterion due to Penrose [25] for the original Vlasov–Poisson equation, gives the existence (resp. the non-existence) of unstable generalized eigenvalues which may imply large time linear instabilities (cf. [10] for this point of view). However for V–D–B equation, due to the homogeneity of the dispersion relation, unstable modes whenever they exist are of the form $\omega(k) = \omega^* k$ with $\Im\omega^* \neq 0$. Hence the relation to prove their existence is not a simple adaptation of the Penrose criterion.

Below explicit examples given in [1] and [2] are recalled to show that the well-posedness of the Cauchy problem depends on the structure of the function $v \mapsto G(v)$. For some initial data the linearized problem may have no solution even in the sense of distributions. This remark extends to the nonlinear case which illustrates the natural connections between the stability of the linearized and the full nonlinear system.

2.1 *The Stability for the One Bump Profile*

To emphasize the role of the "bumps" in the stationary profile we recall below the following

Theorem 1. *Assume that the stationary profile $v \mapsto G(v)$ satisfies for some $a \in \mathbb{R}$, the relation*

$$G'(v) := -H(v)(v-a) \text{ with } \quad H(v) > 0. \tag{3}$$

Then any smooth solution $f(t,x,v)$ of the linearized V–D–B equation

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) - G'(v)\partial_x \rho_f(t,x) = 0, \tag{4}$$

satisfies the energy identity,

$$\frac{d}{dt}\left(\int_{\mathbb{R}\times\mathbb{R}} H^{-1}(v)(f(t,x,v))^2 dxdv + \int_{\mathbb{R}} (\rho(t,x))^2 dx\right) = 0.$$

The proof (cf. [19] and [1]) follows from the basic conservation laws of mass and energy combined with the formula (3). From the above Theorem 1 one deduces the following

Corollary 1. *With the function $v \mapsto G(v)$ as in the Theorem 1, and denoting by $\mathscr{H}$ the Hilbert space of functions f such that*

$$\int_{\mathbb{R}\times\mathbb{R}} H^{-1}(v)(f(t,x,v))^2 dxdv + \int_{\mathbb{R}} (\rho(t,x))^2 dx < \infty,$$

the solutions of the Cauchy problem with initial data $f_0(x,v) \in \mathscr{H}$ are described by a unitary group of operators.

2.2 Synthesis of Plane Waves and Unstable Modes

Plane waves of the form,

$$e_k(t,x,v) = A(k,v)\exp\left(i(kx - \omega(k)t)\right),$$

are solutions of the Eq. (4) whenever they satisfy the dispersion relation

$$(-i\omega(k) + ikv)A(k,v) - ik\left(\int_{\mathbb{R}} A(k,v)dv\right)G'(v) = 0,$$

or with $\omega(k) = \omega^* k$,

$$1 - \int_{\mathbb{R}} \frac{G'(v)}{v-\omega^*} dv = 0. \tag{5}$$

Then for any ω^* solution of (5), the functions

$$f(t,x,v) = \int_{\mathbb{R}} \frac{G'(v)}{v-\omega^*}\, \hat{\rho}(k)\, e^{i(kx - k\omega^* t)}\, dk,$$

are (if they exist) the unique solutions of the linear Cauchy problem with initial data

$$f(0,x,v) = \int_{\mathbb{R}} \frac{G'(v)}{v-\omega^*}\, \hat{\rho}(k)\, e^{ikx}\, dk.$$

As a consequence if there exists a ω^* solution of (5) with $\Im\omega^* \neq 0$, the Cauchy problem is ill-posed in any Sobolev space (because polynomial decreasing of Fourier modes with any speed does not compensate exponential growth).

The following statement, which is an adapted version of the Penrose criterion [25], illustrates the relation between multiple bumps and instabilities.

Theorem 2. *Assume that the stationary profile*

$$v \mapsto G(v) \geq 0, \quad \text{and} \quad \int_{\mathbb{R}} G(v)dv = 1,$$

has a minimum for $v = 0$ and is even (i.e. $G(v) = G(-v)$), then for some $\epsilon > 0$ small enough, there is at least one non oscillatory unstable mode $\omega^* = i\beta$ for the equation linearized near $G_\epsilon = \epsilon^{-1}G(\epsilon^{-1}v)$.

Proof. Introducing the ϵ-dependent continuous function,

$$\beta \mapsto I_\epsilon(\beta) = \int_{\mathbb{R}} \frac{G'_\epsilon(v)}{v - i\beta}dv = \int_{\mathbb{R}} \frac{G'_\epsilon(v)(v + i\beta)}{v^2 + \beta^2}dv = \int_{\mathbb{R}} \frac{G'_\epsilon(v)v}{v^2 + \beta^2}dv,$$

for which one has,

$$I_\varepsilon(\infty) = 0, \quad \text{and} \quad I_\varepsilon(0) = \int_{\mathbb{R}} \frac{G'_\epsilon(v)}{v}dv = \int_{\mathbb{R}} \frac{G'_\epsilon(v) - G'_\epsilon(0)}{v}dv = \int_{\mathbb{R}} \frac{G_\epsilon(v)}{v^2}dv.$$

The last integration by part is justified by the fact that $G'(0) = 0$ and the convergence at $\pm\infty$ of the integral of v^{-2}. Eventually one has

$$I_\varepsilon(0) = \int_{\mathbb{R}} \frac{G_\epsilon(v)}{v^2}dv = \frac{1}{\epsilon^2}\int_{\mathbb{R}} \frac{G(v)}{v^2}dv > 1, \quad \text{for } \epsilon \text{ small enough.}$$

Therefore by continuity there exists at least one $\omega^* = i\beta^*$ solution of the dispersion relation (5). □

2.3 Consequences for the Original Nonlinear V–D–B Equation with General Initial Data

Theorem 3. *Let $\dot{H}^m$ be the space of functions $f \in L^\infty(\mathbb{R}_x, L^1(\mathbb{R}_v))$ with, for $1 \leq \ell \leq m$, derivatives $\partial_x^\ell f \in L^2(\mathbb{R}_x; L^1(\mathbb{R}_v))$.*
For every m, the Cauchy problem for the dynamics $S(t)$ defined by the V–D–B equation is not locally ($\dot{H}^m \mapsto \dot{H}^1$) well-posed.

Proof. Let be

$$\phi(t,x,v) = \int \frac{G'(v)}{v-\omega^*}\,\hat{\rho}(k)\,e^{i(kx-\omega^* kt)}\,dk.$$

A solution $f(t,x,v,s)$, with the initial data

$$f(0,x,v,s) = G(v) + s\phi(0,x,v),$$

for the nonlinear V–D–B equation gives

$$\tilde{f}(t,x,v) = \frac{d}{ds} f(t,x,v,s)_{|s=0},$$

as a solution for the linearized one. But we obtain a contradiction since $\tilde{f}$ is not well defined even in the distributional sense. □

From the structure of the solution

$$f(t,x,v) = \int_{\mathbb{R}} \frac{G'(v)}{v-\omega(k)/k}\,\hat{\rho}(k)\,e^{i(kx-\omega(k)t)}\,dk,$$

with $w(k) = w{*}k$ and $\Im w^* \neq 0$ one observes that the linear problem (and a fortiori the nonlinear one) will be well-posed if the Fourier transform of initial data are exponentially decreasing, which by use of Paley–Wiener Theorem [24] means that initial data must be analytic in a strip. And this is in agreement with the following forerunner result of Jabin and Nouri [17]:

Theorem 4 (Jabin-Nouri 2011). *For any* (x,v) *analytic function* $f_0(x,v)$ *with*

$$\forall \alpha,\ m\ ,\ n, \quad \sup_x |\partial_x^m \partial_v^n f_0(x,v)|(1+|v|)^\alpha = C(m,n)o(|v|),$$

there exists, for a finite time T, an analytic solution of the Cauchy problem for the V–D–B equation.

3 Hamiltonian Structure of the Vlasov Equation and Application to Some Examples

3.1 Hamiltonian Structure

Below we just recall what would be essential for the present contribution. As it is the case for the classical Vlasov–Poisson equation, one may (cf. [22]) start from the conservation of the energy

$$\mathscr{E} = \int_{\mathbb{R}_x^d}\int_{\mathbb{R}_v^d} \frac{|v|^2}{2} f(t,x,v)dxdv + \frac{1}{2}\int_{\mathbb{R}_x^d} (\rho_f(t,x))^2 dx\,, \quad \rho_f(t,x) = \int_{\mathbb{R}_v^d} f(t,x,v)dv.$$

With the introduction of the Gâteaux derivative of this energy

$$\frac{\delta \mathscr{E}}{\delta f} = \frac{|v|^2}{2} + \rho_f(t,x),$$

and of the Poisson bracket

$$\{g,f\} = \nabla_v g \cdot \nabla_x f - \nabla_x g \cdot \nabla_v f,$$

the V–D–B equation is equivalent to the "Hamiltonian system"

$$\partial_t f = \left\{ f, \frac{\delta \mathscr{E}}{\delta f} \right\}. \tag{6}$$

Remark 1. Following Benney [3], we can obtain a new family of invariants for the one-dimensional Vlasov–Dirac–Benney and Vlasov–Poisson equations. To this purpose, let us define the velocity moments of the distribution function f such that

$$\partial_x A_0 = E(t,x) = -\partial_x \phi(t,x), \quad \text{and for} \quad n \geq 1, \;\; A_n[f](t,x) = \int_{\mathbb{R}} v^n f(t,x,v)dv.$$

Velocity integration of V–D–B or V–P equations against polynomial in velocity leads to the moment hierarchy

$$\partial_t A_n + \partial_x A_{n+1} + n\partial_x A_0(x)\partial_x A_{n-1} = 0. \tag{7}$$

Defining the generating function $f(t,x;z)$ such that

$$f(t,x;z) := \sum_{n\geq 0} A_n(t,x) z^n,$$

it can be easily shown that the moment hierarchy (7) is equivalent to an equation on the generating function f, given by

$$z\partial_t f + \partial_x f = (1 - L(zf))\partial_x f(z=0), \tag{8}$$

with $L = z^2\partial_z$. Now rescaling the differential operator L as $\varepsilon z^2 \partial_z$ in (8), using a recursive procedure which involves recursive multiplication of (8) by εz^2 and its z-differentiation, gathering terms of same power in ε and making some summation manipulations (cf. [3]), we obtain the following infinite system of conservation laws

nicolas.besse@univ-lorraine.fr

$$\frac{\partial}{\partial t}\left(\sum_{n\geq 0} L^n\left(\frac{z^{n+1}f^{n+1}}{(n+1)!}\right)\right)+\frac{\partial}{\partial x}\left(\sum_{n\geq 0} L^n\left(\frac{z^n f^{n+1}}{(n+1)!}\right)-f(0)\right)=0. \tag{9}$$

In (9) each power n of z yields a distinct conservation laws and invariant. These new invariants for the one-dimensional V–D–B and V–P equations are thus polynomials of ϕ and velocity moments A_n, $n \geq 1$.

3.2 Zakharov–Grenier Representation and Benney Equation

Observe also that it is always possible to write the solutions of the Vlasov equation on the form (cf. [14, 28])

$$f(t,x,v)=\int_M \rho(t,x,\sigma)\delta(v-u(t,x,\sigma))d\sigma, \tag{10}$$

with $(M, d\sigma)$ a probability space. These notations are consistent with the macroscopic definition of density and momentum, according to the formulas,

$$\rho(t,x)=\int_{\mathbb{R}^d_v} f(t,x,v)dv=\int_M \rho(t,x,\sigma)d\sigma,$$

$$\rho(t,x)u(t,x)=\int_{\mathbb{R}^d_v} vf(t,x,v)dv=\int_M u(t,x,\sigma)\rho(t,x,\sigma)d\sigma.$$

Such decomposition is not unique and depends in particular on the form of this decomposition at time $t = 0$. Moreover a distribution function $f(t,x,v)$ given by (10) is a distributional solution of the V–D–B equation if and only if the functions $\rho(t,x,\sigma)$ and $u(t,x,\sigma)$ are solutions of the system

$$\begin{aligned}&\partial_t\rho(t,x,\sigma)+\nabla_x\cdot(\rho(t,x,\sigma)u(t,x,\sigma))=0,\\&\partial_t\left(\rho(t,x,\sigma)u(t,x,\sigma)\right)+\nabla_x\cdot(\rho(t,x,\sigma)u(t,x,\sigma)\otimes u(t,x,\sigma))\\&\quad+\rho(t,x,\sigma)\nabla_x\int_M \rho(t,x,\sigma)d\sigma=0.\end{aligned} \tag{11}$$

In one space dimension with $(M, d\sigma)$ being respectively the interval $(0, 1)$ and the Lebesgue measure, the system (11) turns out to be the Benney system

$$\begin{aligned}&\partial_t\rho(t,x,\sigma)+\partial_x(\rho(t,x,\sigma)u(t,x,\sigma))=0,\\&\partial_t u(t,x,\sigma)+u(t,x,\sigma)\partial_x u(t,x,\sigma)+\partial_x\int_0^1 \rho(t,x,\sigma)d\sigma=0,\end{aligned} \tag{12}$$

which has been derived by Zakharov from the original Benney equation [3] by using a Lagrangian parametrization (cf. [28]) as a model of water-waves for long waves. Hence the name "Benney" in the title of this contribution.

3.3 Kinetic Representations

Replacing the Lebesgue measure by the counting measure on the discrete set

$$M = \{1, 2, \dots, N \leq \infty\},$$

the formula (10) becomes the multi-kinetic representation

$$f(t, x, v) = \sum_{1 \leq n \leq N} \rho_n(t, x)\delta(v - u_n(t, x)).$$

In particular for $N = 1$ and in any space dimension, the mono-kinetic distribution

$$f(t, x, v) = \rho(t, x)\delta(v - u(t, x)),$$

is a solution of the V–D–B equation if and only if the moments $\rho(t, x)$ and $u(t, x)$ are solutions of an isentropic fluid equations,

$$\partial_t\rho + \nabla_x \cdot (\rho u) = 0\,, \qquad \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x\left(\frac{\rho^2}{2}\right) = 0, \tag{13}$$

while in a one-dimensional space variable the multi-kinetic distribution function $f(t, x, v)$ will be a solution of the V–D–B equation if and only if the unknowns $U = ((\rho_1, \rho_2, \dots, \rho_N), (u_1, u_2, \dots, u_N))$ are solutions of the system of conservation laws,

$$\begin{aligned} &\partial_t\rho_n + \nabla_x(\rho_n u_n) = 0\,, \\ &\partial_t(\rho_n u_n) + \partial_x(\rho_n u_n^2) + \rho_n\partial_x\Big(\sum_{1 \leq l \leq N} \rho_l\Big) = 0. \end{aligned}$$

3.4 Waterbag Representation and Equations

Assume that the density profile $v \mapsto f(t, x, v)$, with $0 \leq f(t, x, v) \leq 1$, has only one bump (say for $v = \nu(t, x)$). Then with

$$\begin{aligned} &f_+(t, x, v) = f(t, x, v) \ \text{ for } \ v \geq \nu(t, x), \qquad \text{and} \\ &f_-(t, x, v) = f(t, x, v) \ \text{ for } v \leq \nu(t, x), \end{aligned}$$

one defines on]0, 1[, two functions $\sigma \mapsto v_\pm(t,x,\sigma)$ according to the formulas

$$f_\pm(t,x,v_\pm(t,x,\sigma)) = \sigma, \quad \text{if} \quad \sigma \le \sup_v f_\pm(t,x,v),$$
$$v_\pm(t,x,\sigma) = 0, \quad \text{otherwise.}$$

As it was observed in [5] the density profile f can be reconstructed according to the standard formula (with Y denoting the Heaviside function)

$$f(t,x,v) = \int_0^1 \delta(v - v_+(t,x,\sigma))\mathrm{Y}(v - \nu(t,x))|\partial_\sigma v_+(t,x,\sigma)|\sigma d\sigma$$
$$+ \int_0^1 \delta(v - v_-(t,x,\sigma))\mathrm{Y}(\nu(t,x) - v)|\partial_\sigma v_-(t,x,\sigma)|\sigma d\sigma,$$

or also as

$$f(t,x,v) = \int_0^1 (\mathrm{Y}(v_+(t,x,\sigma) - v) - \mathrm{Y}(v_-(t,x,\sigma) - v))d\sigma,$$

which is an exact weak solution of the V–D–B equation if and only if

$$\partial_t v_\pm + v_\pm \partial_x v_\pm + \partial_x \int_0^1 (v_+(t,x,\sigma) - v_-(t,x,\sigma))d\sigma = 0. \tag{14}$$

Remark 2. From the formulas (14) one deduces the relations

$$\partial_t(v_+ - v_-) + \frac{(v_+ + v_-)}{2}\partial_x(v_+ - v_-) + (v_+ - v_-)\partial_x \frac{(v_+ + v_-)}{2} = 0,$$
$$\partial_t(\partial_\sigma v_\pm) + v_\pm \partial_x \partial_\sigma v_\pm + (\partial_\sigma v_\pm)\partial_x v_\pm = 0,$$

which imply that the following properties,

$$\forall \sigma \in (0,1) \quad v_-(x,t,\sigma) \le v_+(x,t,\sigma),$$
$$\sigma \mapsto v_+(x,t,\sigma) \text{ is decreasing and } \sigma \mapsto v_-(x,t,\sigma) \text{ is increasing,} \tag{15}$$

are preserved by the dynamics [1, 4, 8].

With the infinite set of σ-dependent densities ρ and velocities u,

$$\rho(t,x,\sigma) = v_+(t,x,\sigma) - v_-(t,x,\sigma), \quad u(t,x,\sigma) = \frac{1}{2}(v_+(t,x,\sigma) + v_-(t,x,\sigma)),$$

the (v_-, v_+)-system (14) is equivalent to the fluid type system,

$$\begin{aligned}
&\partial_t\rho(t,x,\sigma) + \partial_x(\rho(t,x,\sigma)u(t,x,\sigma)) = 0,\\
&\partial_t u(t,x,\sigma) + \partial_x\left(\frac{1}{2}u^2(t,x,\sigma) + \frac{1}{8}\rho^2(t,x,\sigma)\right) + \partial_x\int_0^1 \rho(t,x,a)da = 0\,.
\end{aligned} \tag{16}$$

3.5 Hamiltonian Formulation of Fluid Representations

In fact fluid representations of the V–D–B equation, such as "mono-kinetic" model (13), the Zakharov–Benney model (12) and the waterbag model (16), inherit of the Hamiltonian structure of the V–D–B equation (6), with the energy $\mathscr{E}$ specified by the fluid representation that we choose for the distribution function f. To this purpose we introduce the matrix $\mathscr{J}$ defined by

$$\mathscr{J} = -\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

For the "mono-kinetic" model (13), setting $\mathfrak{m}(t,x) = (\rho(t,x),\ u(t,x))^T$, we obtain the Hamiltonian formulation

$$\partial_t\mathfrak{m} = \{\mathfrak{m},\mathscr{E}\}_{\mathrm{MoK}} := \mathscr{J}\,\partial_x\frac{\delta\mathscr{E}}{\delta\mathfrak{m}},$$

leading to the Poisson bracket structure [22, 23],

$$\{F(\mathfrak{m}), G(\mathfrak{m})\}_{\mathrm{MoK}} = \int_{\mathbb{R}} dx\,\frac{\delta F}{\delta\mathfrak{m}}\mathscr{J}\,\partial_x\frac{\delta G}{\delta\mathfrak{m}}.$$

For the Zakharov–Benney model (12), setting $\mathfrak{m}(t,x,\sigma) = (\rho(t,x,\sigma),\ u(t,x,\sigma))^T$, we obtain the Hamiltonian formulation

$$\partial_t\mathfrak{m} = \{\mathfrak{m},\mathscr{E}\}_{\mathrm{ZB}} := \mathscr{J}\,\partial_x\frac{\delta\mathscr{E}}{\delta\mathfrak{m}},$$

leading to the Poisson bracket structure

$$\{F(\mathfrak{m}), G(\mathfrak{m})\}_{\mathrm{ZB}} = \int_0^1 d\sigma\int_{\mathbb{R}} dx\,\frac{\delta F}{\delta\mathfrak{m}}\mathscr{J}\,\partial_x\frac{\delta G}{\delta\mathfrak{m}}.$$

For the waterbag model (16), setting $\mathfrak{m}(t,x,\sigma) = (\rho(t,x,\sigma),\ u(t,x,\sigma))^T = (v_+ - v_-,\ [v_+ + v_-]/2)^T$, we obtain the Hamiltonian formulation

$$\partial_t\mathfrak{m} = \{\mathfrak{m},\mathscr{E}\}_{\mathrm{WB}} := \mathscr{J}\,\partial_x\frac{\delta\mathscr{E}}{\delta\mathfrak{m}},$$

leading to the Poisson bracket structure

$$\{F(\mathfrak{m}), G(\mathfrak{m})\}_{\mathrm{WB}} = \int_0^1 d\sigma \int_{\mathbb{R}} dx \, \frac{\delta F}{\delta \mathfrak{m}} \mathscr{J} \, \partial_x \frac{\delta G}{\delta \mathfrak{m}}.$$

3.6 Analytic Well-Posedness for Solutions in Fluid Representations

Following Safonov [26], one introduces the Hardy type spaces H^s of x-analytic vector-valued functions $U(t,z,\sigma) = (\rho(t, x+iy, \sigma),\, u(t, x+iy, \sigma))$ defined on the tube $\{(x+iy, \sigma) \in \mathbb{C}^d \times M \,:\, |y_i| < s,\ i = 1, \ldots, d\}$ of the d-dimensional complex plane $\mathbb{C}^d$ with norm,

$$\|U\|_s^2 = \sup_{0 \le |y| \le s,\ \sigma \in M} \left(\int_{\mathbb{R}^d} \left| \left(\mathrm{I} + (-\Delta_x)^{1/2}\right)^{\frac{d}{2}+1} U(x+iy,\sigma) \right|^2 dx \right),$$

and the Banach space $X^{\gamma}_{s_0}$ equiped with the norm,

$$\|U\|^{\gamma}_{s_0} = \sup_{0 \le s + \lambda t < s_0} (s_0 - s - \lambda t)^{\gamma} \|U(t)\|_s,$$

where $\gamma \ge 0$, $s_0 > 0$ and $\lambda > 0$. Eventually one denotes by $\mathscr{B}^{\gamma}_{s_o}(r)$ the ball of radius r in such space, i.e.

$$\mathscr{B}^{\gamma} s_o(r) = \left\{ U \in X^{\gamma}_{s_0} \,;\, \sup_{0 \le s + \lambda t < s_0} (s_0 - s - \lambda t)^{\gamma} \|U(t)\|_s < r \right\}. \tag{17}$$

Observe that, for all the examples in a one-dimensional space variable, from the "general Benney equation" (cf. Sect. 3.2) to the "waterbag" (cf. Sect. 3.4), the Cauchy problem can be written in the form,

$$f(t,x,v) = \int_M \rho(t,x,a)\delta(v - u(t,x,\sigma)) d\sigma,$$

$$U(t,x,\sigma) = (\rho(t,x,\sigma), u(t,x,\sigma)), \quad U(t,\sigma) = U(0,\sigma) + \int_0^t \mathscr{F}(U)(\tau,\sigma) d\tau, \tag{18}$$

where $\mathscr{F}$ is an operator which satisfies the hypothesis of the Safonov version of the Cauchy–Kowalewski Theorem (namely Assumptions 1.1 in [26]). Indeed one has $\mathscr{F}(0) = 0$; For $r > 0$, the correspondence $U \mapsto \mathscr{F}(U)$ is a continuous mapping of $\left\{U \in \mathrm{H}^s \,:\, \|U\|_s^2 < r\right\}$ into $\mathrm{H}^{s'}$ with $0 < s' < s < s_0$; and for any $0 < s' < s < s_0$, and for all $U,\ V \in \mathrm{H}^s$, with $\|U\|_s < r$, $\|V\|_s < r$, we have

nicolas.besse@univ-lorraine.fr

$$\|\mathscr{F}(U) - \mathscr{F}(V)\|_{s'} \leq \frac{C(r)}{s - s'}\|U - V\|_s.$$

Finally this leads to the following

Theorem 5. *For solutions given by the formula (18) there exists $\lambda > 0$ depending only the dimension d, and the constant parameters $s_0 > 0$, $r > 0$, and $\gamma \in (0, 1)$ such that for any initial data*

$$U(0, x, \sigma) = (\rho(0, x, \sigma), u(0, x, \sigma)) \in \mathscr{H}^{s_0}$$

with $\|U(0)\|s_0 < r$, one has on the time interval $(0, \frac{s_0}{\lambda})$ a solution $U(t, x, \sigma) \in \mathscr{H}^{s_0-\lambda t}$, with $\|U(t)\|s_0 - \lambda t<r$ of the corresponding Cauchy problem.

Remark 3. In agreement with the representation formula (18), the Theorem 5 concerns (at variance with the Jabin–Nouri Theorem 4) solutions which are analytic with respect to x and t but which can exhibit singularities in the v variable (Dirac masses, sum of Dirac masses, step or Heaviside functions, etc ...).

4 Entropy and Local-in-Time Stability in Sobolev Spaces

4.1 Energy, Conserved Quantities and Entropies

The energy takes the form

$$\mathscr{E}(f) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} |v|^2 f(t, x, v)dxdv + \frac{1}{2}\int_{\mathbb{R}} (\rho_f(t, x))^2 dx,$$

which is the basic conserved quantity of the V–D–B equation written in the Hamiltonian formalism according to the formula

$$\partial_t f + \left\{\frac{\delta\mathscr{E}}{\delta f}, f\right\} = 0. \tag{19}$$

Obviously the energy $\mathscr{E}(f)$ is not a convex function of f. However with the representation

$$f(t, x, v) = \int_M \rho(t, x, \sigma)\delta(v - u(t, x, \sigma))d\sigma,$$

this energy may become a convex functional of the variable $U(t, x, \sigma) = (\rho(t, x, \sigma), u(t, x, \sigma))$ solution of the system

$$\partial_t U + \partial_x F(U) = 0, \tag{20}$$

where the application $U \mapsto F(U)$ is a twice continuously Gâteaux-differentiable nonlinear unbounded operator in $\mathbb{L}^2(M,d\sigma) := L^2(M,d\sigma) \times L^2(M,d\sigma)$, with domain $\mathscr{D}(F) = \mathbb{L}^2 \cap \mathbb{L}^\infty(M,d\sigma)$.

More generally, the invariants $\eta(f) = \eta(U)$ of the dynamics given by (19) or (20) are characterized by the relation

$$0 = \frac{d}{dt}\int_{\mathbb{R}} \eta(U)dx = \int_{\mathbb{R}} dx\, D_U\eta(U)\partial_t U = \int_{\mathbb{R}} dx\, D_U\eta(U)D_UF(U)\partial_x U, \qquad (21)$$

where the symbol D_U denotes the differential with respect to the variable U. In the classical theory of conservation laws, solutions of (21) are called "conserved quantities" and are associated to the notion of flux according to the formula $D_U\eta(U)D_UF(U) = D_UQ(U)$ which implies the relation

$$D_U^2\eta(U)D_UF(U) = (D_UF(U))^T D_U^2\eta(U),$$

i.e. the fact that $D_U^2\eta(U)$ is a symmetrizer for the conservation law and a positive definite symmetrizer when $u \mapsto \eta(U)$ is a convex function.

Extension of these considerations to the system (20) is the object of the next theorem,

Theorem 6. *Let us consider solutions $(t,x,\sigma) \mapsto U(t,x,\sigma)$ of the system,*

$$\partial_t U + \partial_x F(U) = 0,$$

where the application $U \mapsto F(U)$ is a twice Gâteaux-differentiable local operator (with respect to the variables (t,x) which can be considered as fixed parameters) in $\mathbb{L}^\infty(M,d\sigma)$. For a twice Gâteaux-differentiable function $U \mapsto \eta(U)$ defined on $H^s(\mathbb{R}_x;\mathbb{L}^\infty(M,d\sigma))$ with value in $\mathbb{R}$, the following assertions are equivalent:

i) $D_U\eta(U)D_UF(U) = D_UQ(U)$ is a flux.
ii) $\eta(U)$ is a conserved quantity.
iii) $D_U^2\eta(U)D_UF(U)$ is a symmetric (self-adjoint) operator.

Proof. ii) follows from i) by integration of the relation

$$\partial_t\eta(U) = -\partial_x Q(U).$$

If $\eta(U)$ is a conserved quantity, one has

$$0 = \frac{d}{dt}\int \eta(U(t,x))dx = -\int_{\mathbb{R}} D_U\eta(U)D_UF(U)\partial_x U. \qquad (22)$$

Using Gâteaux-derivative of (22), we get for any vector-valued function V,

$$
\begin{aligned}
0 &= \frac{d}{ds}\left(\int_{\mathbb{R}} D_U\eta(U+sV)D_U(F(U+sV))\partial_x(U+sV)dx\right)_{|s=0}\\
&= \int_{\mathbb{R}} D_U^2\eta(U)[V, D_UF(U)\partial_xU]dx + \int_{\mathbb{R}} D_U\eta(U)D_U^2F(U)[V,\partial_xU]dx\\
&\quad + \int_{\mathbb{R}} D_U\eta(U)D_UF(U)\partial_xVdx,\\
&= \int_{\mathbb{R}} (\partial_xU)^T(D_UF(U))^TD_U^2\eta(U)Vdx + \int_{\mathbb{R}} \{D_U(D_U\eta(U)D_UF(U))\\
&\quad -D_U^2\eta(U)D_UF(U)\}[V,\partial_xU]dx + \int_{\mathbb{R}} D_U\eta(U)D_UF(U)\partial_xVdx,\\
&= \int_{\mathbb{R}} (\partial_xU)^T(D_UF(U))^TD_U^2\eta(U)Vdx - \int_{\mathbb{R}} (\partial_xU)^TD_U^2\eta(U)D_UF(U)Vdx\\
&\quad + \int_{\mathbb{R}} \partial_x(D_U\eta(U)D_UF(U))Vdx + \int_{\mathbb{R}} D_U\eta(U)D_UF(U)\partial_xVdx,\\
&= \int_{\mathbb{R}} (\partial_xU)^T(D_UF(U))^TD_U^2\eta(U)Vdx - \int_{\mathbb{R}} (\partial_xU)^TD_U^2\eta(U)D_UF(U)Vdx,
\end{aligned}
$$

which implies the Lax condition

$$
D_U^2\eta(U)D_UF(U) = (D_UF(U))^TD_U^2\eta(U), \tag{23}
$$

and shows that ii) implies iii). The proof of the assertion "iii) implies i)" is a direct adaptation of the same property for functions depending of a finite number of variables and is done as follows. Assume that $U \mapsto \mathscr{R}(U)$ is a linear operator in $\mathbb{L}^\infty(M, d\sigma)$, with $\mathscr{R}(U) = D_U\eta(U)D_UF(U)$, then define $Q(U)$ by the formula

$$
Q(U) = \int_0^1 \mathscr{R}(sU)(U)ds,
$$

and show with one integration by part and self-adjointness of $D_U\mathscr{R}(U)$, that one has for any vector-valued function V,

$$
\frac{d}{d\tau}Q(U+\tau V)_{|\tau=0} = \mathscr{R}(U)(V),
$$

which explicitly means that $U \mapsto \mathscr{R}(U)$ is the Gâteaux derivative of $U \mapsto Q(U)$. It remains to show that $D_U\mathscr{R}(U)$ is self-adjoint, which follows from the Lax condition (23), the obvious relation: for any vector-valued functions V and W one has

$$
(D_U^2F(U)V)W = (D_U^2F(U)W)V,
$$

and the equation

$$D_U\mathscr{R}(U) = D_U(D_U\eta(U)D_UF(U)) = D_U^2\eta(U)D_UF(U) + D_U\eta(U)D_U^2F(U). \quad \square$$

Remark 4. If we denote by the bracket $\{\cdot,\cdot\}_{\mathscr{S}}$, a generic Poisson bracket with $\mathscr{S} \in \{\text{MoK, ZB, WB}\}$ and whose corresponding definitions are established in Sect. 3.5, from the proof of Theorem 6 we get that

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(U)\,dx = \{\eta(U), \mathscr{E}(U)\}_{\mathscr{S}} = 0,$$

which means that the invariant $\eta(U)$ is an involution.

4.2 Stability of Mono-Kinetic and Multi-Kinetic Solutions

In any space dimension d, the energy

$$\mathscr{E}(f) = \frac{1}{2}\int_{\mathbb{R}_x^d} (|u(t,x)|^2 + \rho(t,x))\rho(t,x)dx,$$

of the isentropic system

$$\begin{aligned} &\partial_t\rho + \nabla_x \cdot (\rho u) = 0, \\ &\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x\left(\frac{\rho^2}{2}\right) = 0, \end{aligned} \tag{24}$$

(i.e. for a mono-kinetic solution $f(t,x,v) = \rho(t,x)\delta(v - u(t,x))$), is strictly convex near any constant state $U_0 = (\rho_0, u_0)$ with $\rho_0 > 0$. Following the classical theory of hyperbolic systems of conservation laws [9, 21], this implies the

Theorem 7. *The Cauchy problem for the system (24) and initial data of the form $U_0+\tilde{U}_0(x)$ with $\tilde{U}_0(x) \in H^s(\mathbb{R}^d)$ and $s > d/2+1$, has for a finite time $(0 < t < T*)$, a unique solution of the form $U_0(t,x) + \tilde{U}(t,x)$ with $\tilde{U}(t,x) \in C(0,T;H^s(\mathbb{R}^d))$.*

On the other hand in a one-dimensional space variable, the parameters of the multi-kinetic representation $U = ((\rho_1,u_1),(\rho_2,u_2),\ldots,(\rho_N,u_N))$ are also solutions of a system of $2N$ conservation laws,

$$\begin{aligned} &\partial_t\rho_n + \partial_x(\rho_n u_n) = 0\,, \\ &\partial_t u_n + \partial_x\Big(\frac{u_n^2}{2}\Big) + \partial_x\Big(\sum_{1\le\ell\le N}\rho_\ell\Big) = 0, \end{aligned}$$

with an energy,

$$\mathscr{E}(f) = \frac{1}{2}\int_{\mathbb{R}_x}\left(\sum_{1\le n\le N}\rho_n(t,x)|u_n(t,x)|^2 + \Big(\sum_{1\le n\le N}\rho_n(t,x)\Big)^2\right)dx, \tag{25}$$

which is not (as observed in details in [6]) always convex near a constant state.

For instance with $N = 2$, it is strictly convex near $(\rho_-, u_-, \rho_+, u_+) = (1, -a, 1, a)$ for $a^2 > 2$, and not convex otherwise. Therefore the Theorem 7 can be extended near constant states which ensure the convexity of $\mathscr{E}(U)$, while for perturbations near other initial states the Cauchy problem is ill-posed in any Sobolev and stability requires analyticity of the initial perturbation as in the Theorem 5.

5 Local-in-Time Stability of the One Bump Profile Solution

As observed in the Sect. 3.4, the evolution of a one bump profile can be described either with the velocity variables $v_\pm(t,x,\sigma)$ or with the fluid variables $U(t,x,\sigma) = (\rho(t,x,\sigma), u(t,x,\sigma))$ according to the equations,

$$\partial_t v_\pm + \partial_x\Big(\frac{v_\pm^2}{2}\Big) + \partial_x\int_0^1 (v_+(t,x,\sigma) - v_-(t,x,\sigma))d\sigma = 0\,, \tag{26}$$

or with

$$\begin{aligned}
&\rho(t,x,\sigma) = v_+(t,x,\sigma) - v_-(t,x,\sigma) \quad \text{and}\\
&u(t,x,\sigma) = (v_+(t,x,\sigma) + v_-(t,x,\sigma))/2,\\
&\partial_t\rho(t,x,\sigma) + \partial_x(\rho(t,x,\sigma)u(t,x,\sigma)) = 0,\\
&\partial_t u(t,x,\sigma) + \partial_x\left(\frac{1}{2}u^2(t,x,\sigma) + \frac{1}{8}\rho^2(t,x,\sigma)\right) + \partial_x\int_0^1 \rho(t,x,a)da = 0.
\end{aligned} \tag{27}$$

This system can be reformulated as

$$\partial_t U + \partial_x F(U) = 0, \tag{28}$$

with $U(\sigma) \mapsto F(U)(\sigma)$ a twice Gâteaux-differentiable operator in $\mathbb{L}^\infty(0,1)$. Moreover the energy of these solutions in the $v_\pm(t,x,\sigma)$ or in the $(\rho(t,x,\sigma), u(t,x,\sigma))$ representation is given by

$$\begin{aligned}
\mathscr{E}(f)(t) =&\ \frac{1}{2}\left(\int_{\mathbb{R}\times\mathbb{R}} v^2 f(t,x,v)dxdv + \int_{\mathbb{R}} \left(\rho_f(t,x)\right)^2 dx\right)\\
=&\ \int_{\mathbb{R}_x}\left[\frac{1}{6}\int_0^1 (v_+^3(t,x,\sigma) - v_-^3(t,x,\sigma))d\sigma\right.\\
&\left. +\frac{1}{2}\left(\int_0^1 (v_+(t,x,\sigma) - v_-(t,x,\sigma))d\sigma\right)^2\right]dx,\\
=&\ \frac{1}{2}\int_{\mathbb{R}}\int_0^1 \left(\rho(t,x,\sigma)u^2(t,x,\sigma) + \frac{1}{12}\rho^3(t,x,\sigma)\right)d\sigma dx\\
&+\frac{1}{2}\int_{\mathbb{R}}\left(\int_0^1 \rho(t,x,\sigma)d\sigma\right)^2 dx. \qquad (29)
\end{aligned}$$

Therefore, equipped with a convex entropy (29), the system (27) has for the Cauchy problem a unique solution according to the

Theorem 8. *For any set of initial data*

$$(\tilde\rho^0(x,\sigma), u^0(x,\sigma)) \in \mathbb{L}^\infty(0,1;H^3(\mathbb{R}_x))$$

there exists a finite time T such that the Cauchy problem

$$\begin{aligned}
&\partial_t\rho(t,x,\sigma) + \partial_x(\rho(t,x,\sigma)u(t,x,\sigma)) = 0,\\
&\partial_t u(t,x,\sigma) + \partial_x\left(\frac{1}{2}u^2(t,x,\sigma) + \frac{1}{8}\rho^2(t,x,\sigma)\right) + \partial_x\int_0^1 \rho(t,x,a)da = 0,\\
&\rho(0,x,\sigma) = C + \tilde\rho^0(x,\sigma), \quad u(0,x,\sigma) = u^0(x,\sigma) \quad \text{with} \quad \rho(0,x,\sigma) \geq c > 0,
\end{aligned} \qquad (30)$$

has a unique solution $(\rho(t,x,\sigma) = C + \tilde\rho(t,x,\sigma) \geq c > 0,\ u(t,x,\sigma))$ with

$$(\tilde\rho(t,x,\sigma), u(t,x,\sigma)) \in L^\infty(0,T;\mathbb{L}^\infty(0,1;H^3(\mathbb{R}_x))).$$

Corollary 2. *Let us consider an initial profile*

$$f^0(x,v) = \Upsilon(v)f_+^0(x,v) + (1-\Upsilon(v))f_-^0(x,v),$$

with $f_+^0(v)$ decreasing and $f_-^0(v)$ increasing such that the functions $(\rho^0(x,\sigma), u^0(x,\sigma))$ given by the formulas,

$$f_\pm(v_\pm(\sigma)) = \sigma, \quad \rho^0(x,\sigma) = v_+(x,\sigma) - v_-(x,\sigma),$$

and

$$u^0(x,\sigma) = (v_+(x,\sigma) + v_-(x,\sigma))/2,$$

satisfy the hypothesis of the Theorem 8. Then the distribution function,

$$f(t,x,v) = \int_0^1 (\Upsilon(v_+(t,x,\sigma) - v) - \Upsilon(v_-(t,x,\sigma) - v))d\sigma, \tag{31}$$

is, for $0 < t < T$ *(with* T *given by the Theorem 8), a solution of the problem*

$$\partial_t f + v\partial_x f - \partial_x \left(\int_{\mathbb{R}} f(t,x,v)dv \right) \partial_v f = 0,$$

with initial data $f(0,x,v) = f^0(x,v)$.

Proof. With $U(t,x,\sigma) = (\rho(t,x,\sigma), u(t,x,\sigma))$, the Cauchy problem (30) can be written according to the formulas

$$\begin{aligned} &\partial_t U + \partial_x F(U) = 0, \\ &F(U) = \begin{cases} \rho(t,x,\sigma)u(t,x,\sigma), \\ \dfrac{1}{2}u^2(t,x,\sigma) + \dfrac{1}{8}\rho^2(t,x,\sigma) + \displaystyle\int_0^1 \rho(t,x,a)da\,. \end{cases} \end{aligned} \tag{32}$$

This system has the convex energy

$$\begin{aligned} \eta(U) = \frac{1}{2}\int_{\mathbb{R}}\int_0^1 (\rho(t,x,\sigma)u^2(t,x,\sigma) + \frac{1}{12}\rho^3(t,x,\sigma))d\sigma dx \\ + \frac{1}{2}\int_{\mathbb{R}}\left(\int_0^1 \rho(t,x,\sigma)d\sigma\right)^2 dx. \end{aligned}$$

Therefore (cf. Theorem 6) $D_U^2\eta(U)$ is a well defined positive symmetrizer and local-in-time estimates can be obtained by considering the expression

$$\partial_x^3(\partial_t U + D_U F(U)\partial_x U = 0),$$

on which we can apply the symmetrizer integral operator $D_U^2\eta(U)$ from the left, and proceeding as in the classical case (cf. [1, 9, 21]) to complete the proof of the Theorem 8.

Now, given the functions $v_\pm(t,x,\sigma)$, the fluid variables $(\rho(t,x,\sigma), u(t,x,\sigma))$ are recovered by the formulas

$$v_\pm(t,x,\sigma) = u(t,x,\sigma) \pm \frac{1}{2}\rho(t,x,\sigma).$$

Obviously one has $v_+(t,x,\sigma) \geq v_-(t,x,\sigma)$ and the equations

$$\partial_t(\partial_\sigma v_\pm) + v_\pm\partial_x(\partial_\sigma v_\pm) + (\partial_\sigma v_\pm)\partial_x v_x = 0,$$

imply that the monotonicity of the functions $\sigma \mapsto v_{\pm}(t,x,\sigma)$ are preserved by the dynamics [1, 4, 8]. Eventually one uses the formula (31) to reconstruct the solution of the V–D–B equation. □

Remark 5. Since the equivalent system (26) has also a convex entropy it can be also diagonalized and this leads to a formulation in term of generalized Riemann invariants giving also a stability result but with well adapted regularity of the initial data with respect to the variable σ. This was done in [4] by N. Besse following a method introduced by Teshukov [27].

Remark 6. With no surprise there is a good agreement between the results for the linearized problem and the nonlinear one. The above theorems concerning the "waterbag" and the "mono-kinetic" equations are the counterpart of the stability results near a one bump profile which is the object of the Theorem 2. In particular for the "mono-kinetic" equation this profile is a Dirac mass. Then direct computation shows that the dispersion relation has no complex value solution [1]. The same remarks is also valid for "multi-kinetic model". For instance with $N = 2$, the dispersion relation for the linearized model is

$$1 = \int_{\mathbb{R}} \frac{G'(v)}{v-\omega} dv = \frac{1}{(a-\omega)^2} + \frac{1}{(a+\omega)^2},$$

which has real solutions if $a^2 > 2$, and complex solutions, i.e. unstable modes, otherwise.

6 Wigner or Semi-Classical Limit of Solutions of the Nonlinear Schrödinger Equation

6.1 Formal Derivations

The connection of the Schrödinger equation with a self-consistent potential to the Vlasov equation, via the Wigner limit, is at present very well documented. For instance for the Schrödinger–Poisson equation, i.e. with a self-consistent defocusing Coulomb type potential of the form

$$\int_{\mathbb{R}^d} \frac{1}{|x-y|^{(d-2)}} |\psi(y)|^2 dy$$

not only (with well adapted initial data) the problem is uniformly well-posed but convergence of the Wigner transform, on an arbitrary large time is proven (cf. for instance [20] or [12]).

For the present discussion one starts with a family $\{\psi_\hbar(t,x,\sigma)\}_{\sigma\in M}$, solution of the following quadratic nonlinear Schrödinger equation

$$i\hbar\partial_t\psi_\hbar(t,x,\sigma) = -\frac{\hbar^2}{2}\Delta_x\psi_\hbar(t,x,\sigma) + \left(\int_M |\psi_\hbar(t,x,\sigma)|^2 d\sigma\right)\psi_\hbar(t,x,\sigma).$$

Given the potential

$$\mathscr{V}_\hbar(t,x) = \int_M |\psi_\hbar(t,x,\sigma)|^2 d\sigma,$$

the time-dependent equation

$$i\hbar\partial_t\theta_\hbar(t,x,\sigma) = -\frac{\hbar^2}{2}\Delta_x\theta_\hbar(t,x,\sigma) + \mathscr{V}_\hbar(t,x)\theta_\hbar(t,x,\sigma),$$

defines by the formula

$$\{\theta_\hbar(t,x,\sigma)\}_{\sigma\in M} = U_\hbar(t)\{\theta_0(t,x,\sigma)\}_{\sigma\in M},$$

a family of unitary operators $U_\hbar(t)$ acting in the space $L^\infty(M\,;L^2(0,T;L^2(\mathbb{R}^d_x)))$. Then one introduces the projection operator

$$K_\hbar(t,x,y) = \int_M \psi_\hbar(t,x,\sigma)\otimes\overline{\psi_\hbar(t,y,\sigma)}d\sigma,$$

with energy

$$\mathscr{E}_\hbar(K_\hbar) = \text{Trace}\left(-\frac{\hbar^2}{2}\Delta_x K_\hbar + \mathscr{V}_\hbar K_\hbar\right) < \infty.$$

The operator $K_\hbar$ is a solution of the Von Neumann–Heisenberg equation,

$$\frac{d}{dt}K_\hbar = -\frac{1}{i\hbar}[H_\hbar, K_\hbar] = -\frac{1}{i\hbar}\left[\frac{\delta\mathscr{E}_\hbar}{\delta K_\hbar}(K_\hbar), K_\hbar\right],$$

with

$$H_\hbar = \left(-\frac{\hbar^2}{2}\Delta_x + \mathscr{V}_\hbar\right).$$

Eventually for the Wigner transform of the Von Neumann–Heisenberg equation, which involves the Weyl symbol $W_\hbar(t,x,v)$ defined by the Wigner transform of $K_\hbar$,

$$W_\hbar(t,x,v) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-iy\cdot v}K_\hbar\Big(t, x+\frac{\hbar}{2}y, x-\frac{\hbar}{2}y\Big)dy,$$

one has "at the formal level" (i.e. assuming all sufficient conditions to pass to the limit) the following convergences as $\hbar \to 0$ (Wigner or semi-classical limit):

$$
\begin{aligned}
&W_\hbar(t,x,v) \longrightarrow W(t,x,v),\\
&\mathscr{E}_\hbar(K_\hbar) \longrightarrow \frac{1}{2}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|v|^2 W(t,x,v)dv + \left(\int_{\mathbb{R}^d} W(t,x,v)dv\right)^2\right)dx,\\
&\partial_t W + v\cdot\nabla_x W - \nabla_x\left(\int_{\mathbb{R}^d} W(t,x,w)dw\right)\cdot\nabla_v W = 0.
\end{aligned}
$$

In order to consider "mixed states" as in [20] and connect with Zakharov–Grenier formula (10), we now assume that the functions $\psi_\hbar(t,x,\sigma)$ can be written as

$$
\psi_\hbar(t,x,\sigma) = a_\hbar(t,x,\sigma)e^{i\frac{S_\hbar(t,x,\sigma)}{\hbar}},
$$

with $a_\hbar$ and $S_\hbar$ "uniformly regular" with respect to $\hbar$, then for the Wigner transform one has:

$$
\begin{aligned}
&\lim_{\hbar\to 0} W_\hbar(K_\hbar(t,x,y))\\
&\quad = \lim_{\hbar\to 0}\int_M d\sigma \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{iv\cdot y} a_\hbar(t,x+\frac{\hbar}{2}y,\sigma)e^{i\frac{S_\hbar(t,x+\frac{\hbar}{2}y,\sigma)}{\hbar}}\\
&\qquad \times \overline{a_\hbar(t,x-\frac{\hbar}{2}y,\sigma)e^{-i\frac{S_\hbar(t,x-\frac{\hbar}{2}y,\sigma)}{\hbar}}}dy\\
&\quad = \int_M |a(t,x,\sigma)|^2\delta(v-\nabla_x S(t,x,\sigma))d\sigma = \int_M \rho(t,x,\sigma)\delta(v-u(t,x,\sigma))d\sigma.
\end{aligned}
$$

Taking "formally" the limit $\hbar \to 0$, one obtains with $\rho = \lim_{\hbar\to 0} a_\hbar \overline{a}_\hbar$ and $u = \lim_{\hbar\to 0}\nabla_x S_\hbar$,

$$
\begin{aligned}
&\partial_t\rho(t,x,\sigma) + \nabla_x\cdot(\rho(t,x,\sigma)u(t,x,\sigma)) = 0,\\
&\partial_t u(t,x,\sigma) + u(t,x,\sigma)\cdot\nabla_x u(t,x,\sigma) + \nabla_x\int_M \rho(t,x,a)da = 0,
\end{aligned}
\tag{33}
$$

which is the Benney or V–D–B equation in the Zakharov–Grenier representation. Moreover, on the other hand, with

$$
\psi_\hbar(t,x,\sigma) = a_\hbar(t,x,\sigma)e^{i\frac{S_\hbar(t,x,\sigma)}{\hbar}}, \quad \text{and} \quad w_\hbar(t,x,\sigma) = \nabla_x S_\hbar(t,x,\sigma),
$$

the equation

$$
i\hbar\partial_t\psi_\hbar(t,x,\sigma) = -\frac{\hbar^2}{2}\Delta_x\psi_\hbar(t,x,\sigma) + \left(\int_M |\psi_\hbar(t,x,\sigma)|^2 d\sigma\right)\psi_\hbar(t,x,\sigma),
$$

is equivalent to the system

$$\partial_t a_\hbar(t,x,\sigma)+w_\hbar(t,x,\sigma)\cdot\nabla_x a_\hbar(t,x,\sigma)+\frac{1}{2}a_\hbar(t,x,\sigma)\nabla_x\cdot w_\hbar(t,x,\sigma) = \frac{i\hbar}{2}\Delta_x a_\hbar(t,x,\sigma)\,,$$
$$\partial_t w_\hbar(t,x,\sigma)+w_\hbar(t,x,\sigma)\cdot\nabla_x w_\hbar(t,x,\sigma)+\nabla_x\int_M a_\hbar(t,x,\sigma)\bar{a}_\hbar(t,x,\sigma)d\sigma=0. \quad (34)$$

Remark 7. The above representation is a variant both of the Madelung transform (where the amplitude $a_\hbar$ is taken real) and of the WKB method which is a Taylor expansion. As a consequence $a_\hbar(t,x,\sigma)$ does not remain real for $t\neq 0$ and x real, while $w_\hbar(t,x,\sigma)=\nabla_x S_\hbar(t,x,\sigma)$ remains real for x real. This representation already appeared in [7] and [13] . It was used by Grenier [15, 16] for the validation of the semiclassical limit. Here we apply it both in the analytical and the Sobolev setting (cf. Theorem 9 and Theorem 10) to validate the formal convergence by proving convenient uniform a priori estimates. With no surprise these estimates are in full agreement with the well-posedness or ill-posedness results given above for the V–D–B equation.

6.2 Convergence Proof for Analytic Initial Data

With the notations introduced in the Sect. 3.6, the counterpart of the Theorem 5 turns out to be the

Theorem 9. *There exists $\lambda>0$, depending only on the dimension d, and the constant parameters $s_0>0$, $r>0$ and $\gamma\in(0,1)$ (and in particular independent of $\hbar$) such that for any*

$$(a_\hbar(0,x,\sigma), w_\hbar(0,x,\sigma)=\nabla_x S_\hbar(0,x,\sigma))\in\mathscr{H}^s, \quad (35)$$

with $||(a_\hbar(0),w_\hbar(0))||_{s_0}<r$, there exists on the time interval $(0,\frac{s_0}{\lambda})$ a solution

$$(a_\hbar(t,x,\sigma), w_\hbar(t,x,\sigma)=\nabla_x S_\hbar(t,x,\sigma))\in\mathscr{H}^{s_0-\lambda t},$$

with $\|(a_\hbar(t),w_\hbar(t)\|_{s_o-\lambda t}<r$, of the problem (34) with initial data (35). Moreover these solutions are uniformly bounded (with respect to $\hbar$) in $\mathscr{H}^{s_0-\lambda t}$, so that they converges, as $\hbar\to 0$, to the solutions of Zakharov–Benney equation (33) given by the Theorem 5.

Proof. First observe that any function $x\mapsto f(x)$ defined for $x\in\mathbb{R}^d$, and which is the restriction of an analytic function $f(x+iy)$, defined for $|y_i|<s$, $(i=1,\dots,d)$, can be represented (with the Payley–Wiener Theorem [24]) by the formula

$$f(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat{f}(\xi)d\xi,$$

with $\hat{f}(\xi)$ decaying exponentially for $|\xi| \to \infty$. Hence the complex conjugate

$$\overline{f(x)} = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\overline{\hat{f}(\xi)}d\xi,$$

is also the Fourier transform of a function with the same exponential decay and therefore can be extended as analytic function in the complex domain according to the formula:

$$f^*(x+iy) = \int_{\mathbb{R}^d} e^{-i(x+iy)\cdot\xi}\overline{\hat{f}(\xi)}d\xi. \tag{36}$$

Of course such extension does not coincide with the complex conjugate of $f(x+iy)$ for $y \neq 0$, but it belongs to the same class (in term of regularity) of analytical functions. With this remark in mind, one introduces the analytic extension $(a_\hbar(t,x+iy,\sigma),\ a^*_\hbar(t,x+iy,\sigma),\ w_\hbar(t,x+iy,\sigma))$ of $(a_\hbar(t,x,\sigma),\ \overline{a}_\hbar(t,x,\sigma),\ w_\hbar(t,x,\sigma))$ and write the system (34) in the equivalent form: for $z = x+iy \in \mathbb{C}^d$,

$$\begin{aligned}
&\partial_t w_\hbar(t,z,\sigma) + w_\hbar(t,z,\sigma)\cdot\nabla_z w_\hbar(t,z,\sigma) + \nabla_z \int_M a_\hbar(t,z,\sigma)a^*_\hbar(t,z,\sigma)d\sigma = 0,\\
&\partial_t a_\hbar(t,z,\sigma) + w_\hbar(t,z,\sigma)\cdot\nabla_z a_\hbar(t,z,\sigma) + \frac{1}{2}a_\hbar(t,z,\sigma)\nabla_z\cdot w_\hbar(t,z,\sigma)\\
&\qquad = \frac{i\hbar}{2}\Delta_z a_\hbar(t,z,\sigma),\\
&\partial_t a^*_\hbar(t,z,\sigma) + w_\hbar(t,z,\sigma)\cdot\nabla_z a^*_\hbar(t,z,\sigma) + \frac{1}{2}a^*_\hbar(t,z,\sigma)\nabla_z\cdot w_\hbar(t,z,\sigma)\\
&\qquad = \frac{-i\hbar}{2}\Delta_z a^*_\hbar(t,z,\sigma).
\end{aligned} \tag{37}$$

With the notations

$$U = \begin{pmatrix} w_\hbar(t,z,\sigma)\\ a_\hbar(t,z,\sigma)\\ a^*_\hbar(t,z,\sigma)\end{pmatrix}, \qquad L_\hbar = \begin{pmatrix} 0\\ \frac{i\hbar}{2}\Delta_z\\ -\frac{i\hbar}{2}\Delta_z\end{pmatrix},$$

and

$$F(U) = -\begin{pmatrix} w_\hbar(t,z,\sigma)\cdot\nabla_z w_\hbar(t,z,\sigma) + \nabla_z\int_M a_\hbar(t,z,\sigma)a^*_\hbar(t,z,\sigma)d\sigma\\ w_\hbar(t,z,\sigma)\cdot\nabla_z a_\hbar(t,z,\sigma) + \frac{1}{2}a_\hbar(t,z,\sigma)\nabla_z\cdot w_\hbar(t,z,\sigma)\\ w_\hbar(t,z,\sigma)\cdot\nabla_z a^*_\hbar(t,z,\sigma) + \frac{1}{2}a^*_\hbar(t,z,\sigma)\nabla_z\cdot w_\hbar(t,z,\sigma)\end{pmatrix}, \tag{38}$$

the system becomes

$$\partial_t U = F(U) + L_\hbar(U),$$

which, using a Duhamel's formula, implies

$$U(t) = e^{tL_\hbar}(U_0) + \int_0^t e^{(t-\tau)L_\hbar} F(U(\tau))d\tau = \Phi(U). \tag{39}$$

Since F is bilinear (and linear with respect to the first-order derivative) and since $e^{tL_\hbar}$ is, for any $\hbar$, a unitary operator in H^s, then for any $0 < s' < s < s_0$, one has

$$\|e^{(t-\tau)L_\hbar}F(U(\tau)) - e^{(t-\tau)L_\hbar}F(V(\tau))\|_{s'} \leq \frac{C}{s-s'}\|U(\tau)-V(\tau)\|_s(\|U(\tau)\|_s + \|V(\tau)\|_s),$$

with C depending only on the dimension and in particular not on $\hbar$. Next one uses the $\|\cdot\|_{s_0}^\gamma$-norm of the Banach space $X_{s_0}^\gamma$, i.e.

$$\|U\|_{s_0}^\gamma = \sup_{0\leq s+\lambda t<s_0} (s_0 - s - \lambda t)^\gamma \|U(t)\|_s,$$

and following Safonov [26] shows that

$$\|\Phi(U) - \Phi(V)\|_{s_0}^\gamma \leq \frac{2^{\gamma+1}C(\|U(\tau)\|_s + \|V(\tau)\|_s)}{\gamma\lambda}\|U(\tau) - V(\tau)\|_{s_0}^\gamma.$$

Hence for λ chosen large enough (with respect to C and r), Φ preserves the ball $\mathscr{B}_{s_0}^0(r)$, and is a contraction in $X_{s_0}^\gamma \cap \mathscr{B}_{s_0}^0(r)$. The rest of the proof follows. □

Remark 8. The above proof is simpler than the forerunner result of Gérard [11], and it also provides an extension to mixed states as considered by Lions and Paul [20]. This is essentially due to the fact that [26] version of the Cauchy–Kowalewski Theorem is very well adapted to the problem in the representation proposed by Grenier [15, 16].

6.3 Convergence Proof for Finite Time with Finite Sobolev Regularity

With no surprise, the stability result for the limit equation should find their counterpart at the level of the convergence. In particular the local-in-time stability in Sobolev spaces has been proven for the mono-kinetic solutions. Mono-kinetic solutions correspond to the ($\hbar \to 0$)-limit of the Wigner transform of a "pure WKB-state", i.e.

$$W_\hbar\left(a_\hbar(t,x)e^{i\frac{S_\hbar(t,x)}{\hbar}}\otimes a_\hbar(t,y)^* e^{-i\frac{S_\hbar(t,y)}{\hbar}}\right)\underset{\hbar\to 0}{\longrightarrow}\rho(t,x)\delta(v-u(t,x)).$$

The validity of such convergence comes from standard uniform estimates due to Grenier [15, 16]. For comparison with the rest of the present contribution, this result is recalled below.

Theorem 10 (Grenier [16]). *Let $s > d/2+2$, let $S^0(x)\in H^s(\mathbb{R}^d)$ and $a^0(x,\hbar)$ be a sequence of functions uniformly bounded in $H^s(\mathbb{R}^d)$. Then there exist $T>0$, and solutions*

$$\psi_\hbar(t,x)=a_\hbar(t,x)e^{i\frac{S_\hbar(t,x)}{\hbar}},$$

to the Cauchy problem

$$i\hbar\partial_t\psi_\hbar=-\frac{\hbar^2}{2}\Delta_x\psi_\hbar+|\psi_\hbar|^2\psi_\hbar,\qquad \psi_\hbar(0,x)=a^0(x,\hbar)e^{i\frac{S^0_\hbar(x)}{\hbar}}.$$

Moreover, $a_\hbar(t,x)$ and $S_\hbar(t,x)$ are bounded in $L^\infty(0,T;H^s(\mathrm{IR}^d))$ uniformly in $\hbar$.

To prove this theorem, Grenier starts from the following system

$$\begin{aligned}
&\partial_t w_\hbar(t,x)+w_\hbar(t,x)\nabla_x w_\hbar(t,x)+\nabla_x(\alpha_\hbar^2(t,x)+\beta_\hbar^2(t,x))=0,\\
&\partial_t\alpha_\hbar(t,x)+w_\hbar(t,x)\cdot\nabla_x\alpha_\hbar(t,x)+\frac{1}{2}\alpha_\hbar(t,x)\nabla_x\cdot w_\hbar(t,x)=-\frac{\hbar}{2}\Delta_x\beta_\hbar(t,x),\\
&\partial_t\beta_\hbar(t,x)+w_\hbar(t,x)\cdot\nabla_x\beta_\hbar(t,x)+\frac{1}{2}\beta_\hbar(t,x)\nabla_x\cdot w_\hbar(t,x)=\frac{\hbar}{2}\Delta_x\alpha_\hbar(t,x),
\end{aligned}\tag{40}$$

which corresponds to the restriction to the real domain of (37) and where $\alpha_\hbar(t,x)$ and $\beta_\hbar(t,x)$ denote respectively the real and imaginary part of $a_\hbar(t,x)$. He observes that this system can be symmetrized by a strictly positive matrix S and this will lead to the standard a priori estimates of hyperbolic systems of conservation laws [9]. In fact the existence of such strictly positive symmetrizer is a consequence of the fact that the mass (i.e. with $\rho_\hbar(t,x)=\alpha_\hbar(t,x)^2+\beta_\hbar(t,x)^2$) and the energy of the system,

$$\frac{1}{2}\int_{\mathbb{R}^d}\left(w_\hbar(t,x)^2+\rho_\hbar(t,x)\right)\rho_\hbar(t,x)dx,$$

are a strictly convex invariants.

Hence in $\mathbb{R}^d$ with $s > d/2+2$, there exists an $\hbar$-independent function $(a_0(0,\cdot),w_0(0,\cdot))$ such that

$$\|(a_\hbar(0,\cdot),w_\hbar(0,\cdot))\|_{H^s}\longrightarrow\|(a_0(0,\cdot),w_0(0,\cdot))\|_{H^s},\quad \text{as }\ \hbar\to 0,$$

and the system (40) has for $t < T(\|(a_\hbar(0,\cdot), w_\hbar(0,\cdot))\|_{H^s})$ a unique solution satisfying the estimate

$$\|(a_\hbar(t,\cdot), w_\hbar(t,\cdot))\|_{H^s} \leq C(\|(a_\hbar(0,\cdot), w_\hbar(0,\cdot))\|_{H^s}).$$

7 Conclusion

The fact that in the V–D–B equation, the operator,

$$f \mapsto E_f = -\partial_x \int f(t,x,v)dv,$$

is local in x and of degree 1 has in the present contribution the following consequences. The well-posedness of the Cauchy problem depends drastically on the initial data and this is related to the convexity of the energy (in a convenient class of solutions).

With such locality the notion of invariants in the sense of Hamiltonian systems, and the notion of conserved quantities for conservation laws do coincide.

The V–D–B equation appears also as the semi-classical or Wigner limit (with $\hbar \to 0$) of solutions of the nonlinear self-consistent Schrödinger equation. Such limit can formally be described and proven in the general case, i.e. for mixed WKB-states initial data only when the initial data are analytic. Otherwise that would be in contradiction with the cases where the limit Cauchy problem is not well-posed. On the other hand it is only for pure WKB-states initial data that the limit (which will be a mono-kinetic solution) is proven with finite Sobolev type regularity.

The above observations remain true in a one-dimensional space variable where not only the nonlinear Schrödinger equation but also its generalization as system of coupled equations (for mixed states) are integrable (cf. Zakharov [28]). This confers to the V–D–B equation a status of quasi-integrable equation with an infinity of invariant quantities, limit of the corresponding invariants at the level of the Schrödinger equations. The above properties being in some sense algebraic, the proof of convergence with analyticity hypothesis seems well adapted to such considerations even if convergence proofs have been obtained (as in the d-dimensional case) for a genuine scalar equation (not a system) either as above by the theorem of Grenier, or in the spirit of scattering theory by Jin, Levermore and McLaughlin [18].

References

1. Bardos, C., Besse, N.: The Cauchy problem for the Vlasov–Dirac–Benney equation and related issues in fluid mechanics and semi-classical limits. Kin. Relat. Models **6**(4), 893–917 (2013)
2. Bardos, C., Nouri, A.: A Vlasov equation with Dirac potential used in fusion plasmas. J. Math. Phys. **53**(11), 115621–115637 (2012)

3. Benney, D.J.: Some properties of long nonlinear waves. Stud. Appl. Math. **52**, 45–50 (1973)
4. Besse, N.: On the waterbag continuum. Arch. Ration. Mech. Anal. **199**(2), 453–491 (2011)
5. Berk, H., Nielsen, C., Robert, K.: Phase space hydrodynamics of equivalent non linear systems: experimental and computational observations. Phys. Fluids **13**(4), 980–985 (1967)
6. Besse, N., Berthelin, F., Brenier, Y., Bertrand, P.: The multi-water-bag equations for collisionless kinetic modeling. Kin. Relat. Models **2**(1), 39–90 (2009)
7. Chazarain, J.: Spectre d'un hamiltonien quantique et mécanique classique. Commun. Partial Differ. Equ. **5**(6), 595–644 (1980)
8. Crandall, M.G., Tartar, L.: Some relations between nonexpansive and order preserving mappings. Proc. Am. Math. Soc. **78**, 385–390 (1980)
9. Dafermos, C.: Hyperbolic Conservation Laws in Continuum Physics. Springer, New York (2000)
10. Degond, P.: Spectral theory of the linearized Vlasov-Poisson equation. Trans. Am. Math. Soc. **294**(2), 435–453 (1986)
11. Gérard, P.: Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire. Séminaire sur les Equations aux Dérivées Partielles 1992–1993, Exp. No. XIII, 13 pp. Ecole Polytech., Palaiseau (1993)
12. Gérard, P., Markowich, P., Mauser, N., Poupaud, F.: Homogenization limits and Wigner transforms. Commun. Pure Appl. Math. **50**(4), 323–379 (1997)
13. Graffi, S., Martinez, A., Pulvirenti, M.: Mean-field approximation of quantum systems and classical limit. Math. Models Methods Appl. Sci. **13**(1), 59–73 (2003)
14. Grenier, E.: Oscillations in quasineutral plasmas. Commun. Partial Differ. Equ. **21**(3–4), 363–394 (1996)
15. Grenier, E.: Limite semi-classique de l'équation de Schrödinger non linéaire en temps petit. C. R. Acad. Sci. Paris Sér. I Math. **320**(6), 691–694 (1995)
16. Grenier, E.: Semiclassical limit of the nonlinear Schrödinger equation in small time. Proc. Am. Math. Soc. **126**(2), 523–530 (1998)
17. Jabin, P.E., Nouri, A.: Analytic solutions to a strongly nonlinear Vlasov. C. R. Math. Acad. Sci. Paris **349**(9–10), 541–546 (2011)
18. Jin, S., Levermore, C.D., McLaughlin, D.W.: The semiclassical limit of the defocusing NLS hierarchy. Commun. Pure Appl. Math. **52**(5), 613–654 (1999)
19. Kruskal, M.D.: Hydromagnetics and the theory of plasma in a strong magnetic field, and the energy principles for equilibrium and for stability. 1960 La théorie des gaz neutres et ionisés (Grenoble, 1959) pp. 251–274. Hermann, Paris; Wiley, New York
20. Lions, P.-L., Paul, T.: Sur les mesures de Wigner. Rev. Mat. Iberoamericana **9**(3), 553–618 (1993)
21. Majda, A.: Compressible fluid flow and systems of conservation laws in several space variables. Applied Mathematical Sciences, vol. 53. Springer, New York (1984)
22. Morrison, P.J.: Hamiltonian and action principle formulations of plasma physics. Phys. Plasmas **12**, 058102 (2005)
23. Morrison, P.J.: Hamiltonian description of the ideal fluid. Rev. Mod. Phys. **70**(2), 467–521 (1998)
24. Paley, R.C., Wiener, N.: Fourier transforms in the complex plane, vol. 19. AMS, USA (1934)
25. Penrose, O.: Electronic instabilities of a non uniform plasma. Phys. of Fluids **3**(2), 258–265 (1960)
26. Safonov, M.: The abstract Cauchy-Kovalevskaya theorem in a weighted Banach space. Commun. Pure Appl. Math. **48**(6), 629–637 (1995)
27. Teshukov, V.M.: On hyperbolicity of long-wave equations. Soviet Math. Dokl. **32**, 469–473 (1985)
28. Zakharov, V.E.: Benney equations and quasiclassical approximation in the inverse problem method. Funktsional. Anal. i Prilozhen. (Russian) **14**(2), 15–24 (1980)

nicolas.besse@univ-lorraine.fr